Let \( \tilde{f} \in C^1(X) \). For \( \beta \in \mathbb{R}, s, \varepsilon > 0 \) we set

\[
\tilde{f}_{(\beta)} = \{ x \in X : \tilde{f}(x) < \beta \} \\
K_{(\beta)} = \{ x \in X : \tilde{f}(x) = \beta, \tilde{f}'(x) = 0 \} \\
N_{(\beta, s)} = \{ x \in X : \| \tilde{f}(x) - \beta \| < s, \| \tilde{f}'(x) \| < \varepsilon \}
\]

\( \mathcal{U}(\beta, s) \)

**Lemma 6.3:** Suppose \( \tilde{f} \) satisfies \( (P.S.)_{\beta} \). Then:

(i) \( K_{(\beta)} \) is compact

(ii) \( (\mathcal{U}_{(\beta, s)}) \) and \( (\mathcal{U}_{(\beta, s)}) \) are fundamental systems of open neighborhoods of \( K_{(\beta)} \), i.e., given any neighborhood \( N \) of \( K_{(\beta)} \in \mathcal{S} \times 0, \varepsilon > 0 \):

\[
N = N_{(\beta, s)} \Rightarrow N = \mathcal{U}_{(\beta, s)}.
\]

(iii) If \( K = \emptyset \), there exists \( s > 0 \) such that \( \mathcal{U}_{(\beta, s)} = \emptyset \),
Proof:

(i) \( (u_k) \in K_\beta \) is obviously a \((P, S)_{\beta}\)-sequence.

\[ u_k \rightarrow u \in K_\beta \quad \text{by} \ (P, S)_{\beta} \quad \text{for} \quad \text{a subsequence}. \]

(ii) Assume that the statement was wrong, i.e., there is a neighborhood \( V \) of \( K_\beta \) such that \( V, s \ \setminus \ \phi \) for all \( s > 0 \).

\[ \Rightarrow \exists \ u_k \in V \ \setminus \ \phi \]

\[ \Rightarrow (u_k) \text{ is } (P, S)_{\beta}-\text{seq.} \]

\[ \Rightarrow \text{for a subsequence } u_k \rightarrow u \in K_\beta \quad \text{c.}\]

Hence, \( u_k \in V \) for \( k \) large.

If \( V, s \ \setminus \ \phi \) for every \( s > 0 \), let \( u_k \in K_\beta \) for \( k \).

Let \( \sqrt{x} \in K_\beta \) with

\[ \|u_k - \sqrt{x}\|_\beta \leq \frac{1}{k} \]
for a subsequence, \( v_k \to u \) and thus \( u_k \to u \).

Hence, \( u_k \in V \) for \( k \) large.

\( V \neq \emptyset \) is a neighborhood of \( x_0 \).

---

**Theorem 6.10:** \( \text{THE deformation lemma} \)

Let \( \overline{S} \in C^1(x) \) sat. \( (P-S) \) for a given \( \beta > 0 \) and let \( V \) be a neighborhood of \( x_0 \). Then there exists \( 0 < \varepsilon < \overline{S} \) and \( \overline{S} \in C^1(x \times [0,1]) \) s.t.

1. \( \overline{S}(\cdot, t) : X \to X \) is a homeomorphism for \( 0 < t < T \).
2. \( \overline{S}(u, t) = u \) if \( t = 0 \) or \( \overline{S}(u) = 0 \) or \( |\overline{S}(u) - \beta| \geq \varepsilon \).
3. \( t \to \overline{S}(\overline{S}(u), t) \) is monotonically increasing for \( t \in X \).
4. \( \overline{S}(\overline{S}(x), t + \varepsilon) \subseteq \overline{S}(x) \cap B_{\beta+\varepsilon} \) for \( \varepsilon > 0 \).
Proof. (i) It is a neighborhood of \( y \).

By \( \delta, \varepsilon > 0 \) such that
\[
\forall y, \varepsilon, \delta < \delta \begin{align*}
\exists \beta, \gamma \in \mathbb{R} & : y \in \mathcal{V}(\beta, \gamma, \delta, \varepsilon) \\
\beta & > 0, \gamma > 0, \delta > 0
\end{align*}
\]

Fix \( \delta, \varepsilon > 0 \) such that
\[
\forall y, \varepsilon, \delta < \delta \begin{align*}
\exists \beta, \gamma \in \mathbb{R} & : y \in \mathcal{V}(\beta, \gamma, \delta, \varepsilon) \\
\beta & > 0, \gamma > 0, \delta > 0
\end{align*}
\]

Thus
\[
\varepsilon = \frac{1}{4} \min \{ \varepsilon, \delta \}
\]

and choose \( \tilde{\gamma} \in C^\infty(\mathbb{R}) \) with
\[
\tilde{\gamma}(x) = \begin{cases} 
1 & 1 - \beta \leq x \\
0 & 1 - \beta \geq 2 \varepsilon
\end{cases}
\]

and \( \eta : X \to \mathbb{R} \) Lipschitz with \( 0 \leq \eta \leq 1 \) and
\[
\eta(u) = \begin{cases} 
1 & u \in \mathcal{V}(\beta, \delta, \varepsilon) \\
0 & u \notin \mathcal{V}(\beta, \delta, \varepsilon)
\end{cases}
\]

Set
\[
\varepsilon(u) = -\tilde{\gamma}(k(x)) \cdot \eta(u) \cdot \tilde{\gamma}(u), \quad u \in X
\]

Afterwards we proceed as in Lemma only to show (iv) is a bit more complicated.
We have for \( u \in \mathcal{F}_{\beta + \varepsilon} \) with \( \overline{\mathcal{F}}(\phi(u, t)) \geq \beta - \varepsilon \)

\[
\mathcal{F}(\phi(u, t)) = \mathcal{F}(u) + \int_0^t \mathcal{F}(\phi(u, s)) \, ds
\]

\[
\leq \beta + \varepsilon - \frac{1}{2} \left( \mathcal{F}(\phi(u, +)) - \mathcal{F}(\phi(u, t)) \right)
\]

\[
\leq \beta - \varepsilon - \mathcal{L}^1 \left( \{ t : \phi(u, t) \notin \beta, \varepsilon \} \right)
\]

\[
\leq \frac{\varepsilon}{2}
\]

If either \( u \notin \mathcal{V} \) or \( \phi(u, t) \notin \mathcal{V} \)
we get

\[
\mathcal{L}^1 \left( \{ t : \phi(u, t) \notin \mathcal{V}, \beta, \varepsilon \} \right) \geq \frac{\varepsilon}{2}
\]

Since \( \beta - \varepsilon \) have distance

\[
\mathcal{F}(\phi(u, t)) < \beta + \varepsilon \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
Theorem 6.11: Assume $\bar{F} \in C^1(X)$ admits a relative minimizer $u_0 \in X$ and there exists $u_\lambda \in X$ with $\bar{F}(u_\lambda) < \bar{F}(u_0)$.

Let

$$\beta = \inf \sup_{\gamma \in \Gamma} \bar{F}(\gamma(s))$$

where

$$\Gamma = \{ \gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_\lambda \}$$

and suppose that $\beta$ sats. $(P-S)_\beta$.

Then either

(i) $\beta > \bar{F}(u_0)$ and $X_\beta \neq \emptyset$

(ii) $\beta = \bar{F}(u_0)$ and there is a $u \in X$ which is not a rel. minimizer

(iii) $\beta = \bar{F}(u_0) = \bar{F}(u_\lambda)$ and $u_0, u_\lambda$ can be connected in any neighborhood of $X_\beta$.

**Proof:**

(i) If $\beta > \bar{F}(u_0)$, then $u_0$ & $u_\lambda$ are in different components of $X_\beta$.

apply Theorem.
ii) Suppose $\tilde{F}(u_0) = \beta$ and that $\mathcal{N}_\beta$ consists entirely of relative minimizers.

$\forall u \in \mathcal{N}_{\beta}$, there exists an open neighborhood $V(u)$ such that

$$\tilde{F}(v) \leq \tilde{F}(u) = \beta \quad \forall v \in V(u)$$

For any given neighborhood $V$ of $\mathcal{N}_{\beta}$, we set

$$\tilde{V} := \bigcap_{u \in \mathcal{N}_{\beta}} V(u) \cap V$$

$\tilde{V} \subset V$ is an open neighborhood of $\mathcal{N}_{\beta}$ with

$$\tilde{F}(v) \leq \beta \quad \forall v \in \tilde{V}.$$

Now we apply Theorem 6.16 to $\tilde{V}$ with $\varepsilon = 1$ to get $\tilde{F} \in C^0(X \times C_{[0,1]}X)$ and $\varepsilon \in (0, 1)$ with all the desired properties. Let $y \in \tilde{V}$ be such that

$$\sup_{0 \leq s \leq 1} \tilde{F}(y(s)) < \beta + \varepsilon$$

and

$$y_t := \tilde{F}(y, t).$$
Then
\[ y_1 \in \mathcal{F}_{\beta - \varepsilon} \cup \mathcal{N} \]
and \( y_1(0) = u_0 \) since \( u_0 \in \beta \).
As \( \mathcal{F} \cap \mathcal{N} \) and \( \mathcal{N} \) are disjoint, we get
\[ y_1 \in \mathcal{N} \]
and if furthermore \( u_0 \) was \( u_\gamma \in \mathcal{E} \) and hence \( y_1(x) = u_\gamma \), the Thm. would be proven.

Claim: \( u_\gamma \) is critical.

Proof: If not, we can find a point \( u_\gamma \) near by \( \gamma \) such that
\[ \mathcal{E}(u_\gamma) \leq \mathcal{F}(u_\gamma) \leq \beta = \mathcal{E}(0) \]
but still
\[ \beta = \inf \sup_{\gamma \in \mathcal{E}} \mathcal{E}(\gamma(s)) \]
where \( \mathcal{E} = \{ \gamma \in C^0([0,1],X) : \gamma(0) = u_0, \gamma(1) = \bar{u}_\gamma \} \).

Theorem 6.10
\[ \exists \varepsilon > 0 \text{ with all the props.} \]
with \( \varepsilon = \beta - \mathcal{F}(u_\gamma) \). Take \( \gamma \in \mathcal{E} \) with \( \sup_{\gamma \in \mathcal{E}} \mathcal{E}(\gamma(s)) < \beta + \varepsilon \)
\[ \gamma = \mathcal{E}(\gamma, 1) \in \mathcal{F}_{\beta - \varepsilon} \cup \mathcal{N} \] but
\[ y_1(0) \in \mathcal{N}, \quad y_1(1) \in \mathcal{F}_{\beta - \varepsilon} \]
**Def:** Let $K$ be a topological space, $\Phi \in C^0(K \times [0,1], K)$, $\Phi(\cdot, 0) = \text{id}$. A set $\Omega \in \mathcal{P}(K)$ or a family of sets $\Omega$ is $\Phi$-invariant if
\[
\Phi(f(t)) \in \Omega \quad \forall f(t) \in \Omega, \quad t \in [0,1].
\]

**Examples:**

(i) $\Omega = \{K\}$ is $\Phi$-invariant, whenever $\Phi(\cdot, +) : K \to K$ is injective.

(ii) $\Omega = \{\{u\} : u \in K\}$

(iii) Let $\Phi_0 \in C^0(X)$ be such that $\Phi_0$ has two connected comp. $u_1, u_0$ as different comp. Let
\[
x = \max \{\Phi_0(u_0), \Phi_0(u_1)\} < 0
\]
and
\[
\Pi = \{y \in C^0([0,1]; X) : y(0) = u_0, y(x) = u_1 \}.
\]

Then $\Pi$ is invariant w.r.t. any $\Phi \in C^0(X \times [0,1]; X)$ with $\Phi(\cdot, 0) = \text{id}$ and
\[
\Phi_0(\cdot, t) = u_1 \quad \forall u \in \Phi.
\]
vi) Let $\mathcal{F} \in C^1(\mathbb{R})$, $\alpha < \beta$ and suppose that

$$\beta = \inf \sup_{\gamma \in \Pi_k} \mathcal{F}(\gamma(x))$$

where

$$\Pi_k := \{ \gamma \in C^0(\overline{B_k(0, \mathbb{R}^k) }, \mathbb{R}) : \gamma \big|_{\partial B_k} = y_0 : \partial B_k \rightarrow \mathbb{R} \alpha \}$$

Then $\Pi_k$ is $\mathcal{F}^k$ invariant w.r.t $\mathcal{F}^k$.
Theorem 6: (Minimax principle, Palais’)

Let \( F \in C^1(\mathbb{R}) \) sat. (P-S)

for every \( \beta \in \mathbb{R} \) and let \( \alpha \) be \( E \)-cyclic for all \( E \) satisfying properties (i) - (iii) of Thm.

Let \( \beta \alpha := \inf \sup \frac{F}{A} \neq 0 \bigwedge \lambda \) \( \lambda \in \mathbb{R} \).

If \( |\beta \alpha| < \infty \) we get \( \lim_{\lambda \to \infty} \beta \alpha + \phi \).

Proof: Assume \( |\beta \alpha| < \infty \) but \( \kappa = \phi \). Let \( \varepsilon > 0 \), \( \phi \in C^0(\mathbb{R} \times [0,1]) \).

\( \beta \alpha \) be as in Thm. for \( \varepsilon = 1 \), \( N = \phi \). As \( |\beta \alpha| < \infty \), there is an \( \overline{A} \in \mathbb{R} \) with

\[ \sup \frac{F}{A} < \beta \alpha + \varepsilon \]

\( \overline{A} \triangleq \overline{A} \subset \mathbb{R} \). \( \Delta \alpha \leq \mathbb{R} \) but \( \sup \frac{F}{A} < \beta \alpha - \varepsilon \).