§ 7 Limit cases for the Palais-Smale condition

We have seen in an exercise, that

$$\mathcal{F}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx$$

for $p = 2^* = \frac{2n}{n-2}$ does not satisfy $(P.-S)_\beta$ for all $\beta \in \mathbb{R}$ on $W_0^{1,2}(\Omega)$.

The next Lemma tells us though, that $\mathcal{F}$ does satisfy $(P.-S)_\beta$ for β small. The precise bound on β can be expressed using the best Sobolev-constant

$$S := S(\mathbb{R}^n) = \inf_{\substack{u \in W_0^{1,2}(\mathbb{R}^n) \\ u \not\equiv 0}} \frac{\|\nabla u\|^2_{(\Omega)}}{\|u\|^2_{L^{2^*}(\Omega)}}.$$

Lemma 7.1: Let $\Omega \subset\subset \mathbb{R}^n$, $n \geq 3$. Then any $(P-S)_\beta$ sequence with $\beta < \frac{1}{n} S^{\frac{n}{2}}$ of

$$\mathcal{F}(u) := \frac{1}{2} \int |\nabla u|^2 - \frac{1}{2^*} \int |u|^{2^*}$$

is relatively compact. //

Proof:

(i) $\boxed{\text{Boundedness of } (u_m):}$ We have

$$o(1)\left(1 + \|u_m\|_{\overset{\circ}{W}{}^{1,2}}\right) + \frac{2}{n} S^{n/2} = 2\, \mathcal{F}_\lambda(u_m) - \langle u_m, \mathcal{F}'(u_m) \rangle$$

$$= \left(1 - \frac{2}{2^*}\right) \int_\Omega |u_m|^{2^*} dx$$

where $o(1) \to 0$ as $m \to \infty$. Hence,

$$\|u_m\|^2_{\overset{\circ}{W}{}^{1,2}} = 2\, \mathcal{F}(u_m) + \frac{2}{2^*} \int_\Omega |u_m|^{2^*} dx$$

$$\leq C + o(1) \|u_m\|_{\overset{\circ}{W}{}^{1,2}}$$

and thus

$\|u_m\|^2_{\overset{\circ}{W}{}^{1,2}}$ is bounded.

(ii) We may assume that $u_m \to u$ weakly in $\overset{\circ}{H}{}^{1,2}(\Omega)$ and strongly in L^p for all $p < 2^*$ (by Rellich Kondrakov). Hence,

$$0 \overset{m\to\infty}{\leftarrow} \langle \varphi, \mathcal{F}'(u_m) \rangle = \int_\Omega \left(\nabla u_m \nabla \varphi - |u_m|^{2^*-2} u_m \varphi \right) dx$$

$$\overset{m\to\infty}{\longrightarrow} \int \left(\nabla u \nabla \varphi - |u|^{2^*-2} u \varphi \right) dx$$

$$= \langle \varphi, \mathcal{F}'(u) \rangle = 0. \quad \forall \varphi \in C_c^\infty(\Omega)$$

Testing this with $\varphi = u$ we get

$$0 = \langle u, \mathcal{F}'(u) \rangle = \int_\Omega \left(|\nabla u|^2 - |u|^{2^*} \right) dx$$

and hence

$$\mathcal{F}(u) = \left(\frac{1}{2} - \frac{1}{2^*} \right) \int_\Omega |u|^{2^*} dx \geq 0.$$

From Lemma we know that

$$\int |\nabla u_m|^2 dx = \int_\Omega |\nabla (u_m - u)|^2 dx + \int_\Omega |\nabla u|^2 dx + o(1)$$

and

$$\int_\Omega |u_m|^{2^*} dx = \int_\Omega |u_m - u|^{2^*} dx + \int |u|^{2^*} dx + o(1)$$

so that

$$\mathcal{F}(u_m) = \mathcal{F}(u) + \mathcal{F}(u_m - u) + o(1)$$

Since furthermore

$$\int |u_m|^{2^*-2} u_m (u_m - u)\, dx$$

$$= \int \left(|u_m|^{2^*} - |u|^{2^*} dx \right) dx$$

$$+ \underbrace{\int \left(-|u_m|^{2^*-2} u_m + |u|^{2^*-2} u \right) u \, dx}_{= o(1)}$$

$$= \int_\Omega |u_m - u|^{2^*} dx + o(1)$$

we get

$$o(1) = \langle u_m - u, \mathcal{F}'(u_m) \rangle$$

$$= \langle u_m - u, \mathcal{F}'(u_m) - \mathcal{F}'(u) \rangle$$

$$= \int_\Omega \left(|\nabla (u_m - u)|^2 - |u_m - u|^{2^*} \right) dx + o(1)$$

So

$$\mathcal{F}(u_m - u) = \underbrace{\left(\frac{1}{2} - \frac{1}{2^*} \right)}^{\frac{1}{2} - \frac{n-2}{2n} = \frac{1}{n}} \int_\Omega |\nabla (u_m - u)|^2 dx$$

$$+ o(1)$$

$$\rightsquigarrow \quad \mathcal{F}(u_m - u) = \mathcal{F}(u_m) - \mathcal{F}(u) + o(1)$$

$$\leq \mathcal{F}(u_m) + o(1) \leq c < \frac{1}{n} S^{n/2}$$

So

$$\| u_m - u \|^2_{W^{1,2}} \leq c < S^{n/2}$$

and hence $\quad \|u_m - u\|^{2^*}_{L^{2^*}} \leq S^{-1} \|u_m - u\|^2_{W^{1,2}_0}$

Hence,

$$\| u_m - u \|^2_{W^{1,2}_0} \underbrace{\left(1 - S^{-2^*/2} \|u_m - u\|^{2^*-2}_{W^{1,2}_0}\right)}_{\geq c > 0}$$

$$\leq \int \left(|\nabla(u_m - u)|^2 - |u_m - u|^{2^*} \right) dx$$

$$= o(1).$$

and thus $\quad u_m \to u$

$$\left(2^*/2 = \frac{n}{n-2} \right.$$

$$2^* - 2 = \frac{2n - 2n + 4}{n-2}$$

$$= \frac{4}{n-2}$$

Lemma 7.2: Suppose (v_m) is a $(P.S)_\beta$-sequence for $\mathcal{F}$ in $W_0^{1,2}(\Omega)$. Then
$\exists\, k \in \mathbb{N}_0$, $R_m^j \in (0,\infty)$, $x_m^j \in \Omega$ with

$$R_m^j \xrightarrow{m\to\infty} \infty \quad \forall\, 1 \le j \le k$$

and a solution $u_0 \in \overset{\circ}{W}{}^{1,2}(\Omega)$ of (7.1) and non-trivial sol. $u_j \in \overset{\circ}{W}{}^{1,2}(\mathbb{R}^n)$ to

$$-\Delta u = |u|^{2^*-2}u \quad \text{on } \mathbb{R}^n$$

such that

$$\Big\| u_m - u_0 - \sum_{j=1}^{k} u_m^j \Big\|_{\overset{\circ}{W}{}^{1,2}(\mathbb{R}^n)} \to 0$$

Here,

$$u_m^j(x) = \left(R_m^j\right)^{\frac{n-2}{2}} u^j\left(R_m^j(x - x_m^j)\right)$$

Moreover,

$$\mathcal{F}(u_m) \to \mathcal{F}(u^0) + \sum_{j=1}^{k} \mathcal{F}(u^j)$$

//

Remark:

① Gidas and Spruck showed that that the only solutions $u \geq 0$ of (7.) have the form

$$u_\varepsilon^*(x) = \frac{(n(n-2)\varepsilon^2)^{n-2/4}}{(\varepsilon^2 + |x|^2)^{(n-2)/4}}$$

② For a general solution ~~u~~ of we decompose

$$v = v_+ + v_-;$$

where $v_\pm = \pm \max\{\pm u, 0\}$.
Testing the equation with $v_\pm$ we get

$$0 = \int_{\mathbb{R}^n} (-\Delta v - |v|^{2^*-2} v) v_\pm \, dx$$

$$= \int_{\mathbb{R}^n} (|\nabla v_\pm|^2 - |v_\pm|^2) \, dx$$

$$\geq (1 - S^{-2^*/2} \|v_\pm\|_{\mathring{W}^{1,2}}^{2^*-2}) \|v_\pm\|_{\mathring{W}^{1,2}}^2$$

$v_+ \neq 0 \neq v$
$\rightsquigarrow$

$$\mathcal{F}(v_\pm) = \frac{1}{n} \|v_\pm\|_{\mathring{W}^{1,2}}^2$$

$$\geq \frac{1}{n} S^{n/2} =: \beta^*$$

$\rightsquigarrow \quad \mathcal{F}(v) = \mathcal{F}(v_+) + \mathcal{F}(v_-) > 2\beta^*$

③ In the situation of Thm
we hence get

$$\mathcal{F}(u^j) \in \{\beta^*\} \cup]2\beta^*, \infty[.$$

if u^j has
a sign

If we assume that (7.) does not have a non-trivial solution, we hence get that $\mathcal{F}$ satisfies $(P.S.)_\beta$ for all $\beta \in (\beta^*, 2\beta^*)$.

The effect of Topology

We know from Pohozaev's identity that the problem

$$-\Delta u = |u|^{2^*-2} u \quad \text{in } \Omega$$

(7.   )

$$u = 0 \quad \text{on } \partial\Omega$$

has no non-trivial solution on star-shaped Ω. The situation dramatically changes if we look at more complex domains:

Example: Let $\Omega = B_R(0) \setminus B_r(0)$. Then there is a smooth $u \in C^\infty(\bar{\Omega}), u \neq 0$ solving (7.   )

Proof: To get this solution, we minimize

$$\mathcal{F}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx$$

with

$$M := \left\{ u \in W^{1,2}_{rad,0}(\Omega) : \int_\Omega |u|^p = 1 \right\}$$

where

$$W^{1,2}_{rad,0}(\Omega) = \left\{ u \in W^{1,2}_0(\Omega) : u(x) = u(|x|) \right\}.$$

(i) For $u \in W^{1,2}_{rad,0}(\Omega) \cap C^\infty_c(\Omega)$

we get
$$|u(x)|^2 = \int_{|x|}^{\infty} \frac{d}{d\rho} |u(\rho)|^2 \, dr$$
$$= \int_{|x|}^{\infty} |u(\rho)| \cdot |\nabla u(\rho)| \, d\rho$$
$$= C_n \int_{\Omega} \rho^{-n+1} |u(x)| \, |\nabla u(x)| \, dx$$
$$\leq C_n \, r^{-n+1} \left(\|u\|^2_{W^{1,2}_0(\Omega)} \right);$$

So $\|u\|_{L^\infty} \leq C_n r^{-n+1} \|u\|_{W^{1,2}(\Omega)}$

for all $u \in W^{1,2}_{rad,0}(\Omega) \cap C^\infty_c(\Omega)$

Approximation ⇝ argument this also holds for all $u \in W^{1,2}_{rad,0}(\Omega)$

(ii) Let now (u_k) be a bounded sequence in $W^{1,2}_{rad,0}(\Omega)$.
Since $W^{1,2}_{rad,0}(\Omega)$ ~~embeds~~ embeds compactly into $L^2(\Omega)$ we get
$$u_k \to u \quad \text{in } L^2(\Omega)$$
for a subsequence.

For $2 < p < \infty$ we get

$$\int |u_k - u|^p \, dx \le \underbrace{\| u_k - u \|_{L^\infty}^{p-2}}_{\le C} \cdot \int |u_k - u|^2 \, dx \to 0$$

as $k \to \infty$.

Hence the embedding

$$W^{1,2}_{rad,0}(\Omega) \to L^p(\Omega)$$

is compact for any $1 \le p < \infty$.

(iii) The proof now follows exactly as the proof of

We want to conclude the lecture with the sketch of the proof of the following

Theorem 7. 3 (Coron, 1984)

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying

$$\Omega \supset \{ x \in \mathbb{R}^n : R^{-1} < |x| < R \}$$

$$\bar{\Omega} \not\supset \{ x \in \mathbb{R}^n : |x| < R^{-1} \}.$$

If $R \geq 1$ is large enough, then 7. has a positive solution. //

Remark: Theorem 7.3 is a special case of the following fact proven by Bahri & Coron:

Suppose Ω is a domain in $\mathbb{R}^n$ with

$$H_d(\Omega, \mathbb{Z}_2) \neq 0$$

for some $d \geq 1$. Then (7.1) admits a positive solution. //

Sketch of the proof:

(i) For $\sigma \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$

$t \in [0, 1)$:

$$u_t^\sigma(x) = (1-t)^{-\frac{n-2}{2}} \left(\frac{1}{1 + \left| \frac{x-\sigma}{1-t} + \sigma \right|^2} \right)^{\frac{n-2}{2}}$$

$$= \left(\frac{1-t}{(1-t)^2 + |x - t\sigma|^2} \right)^{\frac{n-2}{2}}$$

$\leadsto$ S is attained for u_t^σ, $|u_t^\sigma|^{2^*} dx \overset{w^*}{\rightharpoonup} \left(\int_{\mathbb{R}^n} \left(\frac{1}{1+x^2} \right)^{\frac{n-2}{2}} \right)^2 \cdot \delta_{\{x=\sigma\}}$

and $$u_0^\sigma(x) = \frac{1}{(1+|x|^2)^{(n-2)/2}}.$$

(ii) For $R \geq 1$ we pick a $\varphi_R \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \varphi_R \leq 1$, $\varphi \equiv 1$ on $B_R(0) - B_{R^{-1}}(0)$

$\varphi_R \equiv 0$ on $\mathbb{R}^n - B_{2R}(0)$ and on $B_{R^{-1}/2}(0)$

and

$$|\nabla \varphi_R| \leq \frac{C}{R} \quad \text{on } B_{2R} - B_R$$

$$|\nabla \varphi_R| \leq CR \quad \text{on } B_{R^{-1}} - B_{R^{-1}/2}$$

and set

$$w_t^\sigma = u_t^\sigma \cdot \varphi_R$$

We calculate

$$\int_{\mathbb{R}^n} |\nabla (w_+^\sigma - v_+^\sigma) |^2 \leq$$

$$\leq \int_{(\mathbb{R}^n \setminus B_R) \cup B_{(R)^{-1}}} |\nabla u_+^\sigma|^2 \, dx$$

$$+ c \cdot R^{-2} \int_{B_{2R} - B_R} |u_+^\sigma|^2 \, dx$$

$$+ c \, R^2 \int_{B_{R^{-1}}} |u_+^\sigma|^2 \, dx \to 0$$

uniformly in $\sigma \in \Sigma^n$, $0 \leq t < 1$

We set

$$v_+^\sigma = \frac{w_+^\sigma}{\| w_+^\sigma \|_{L^{2^*}}}$$

and get

$$\| \nabla v_+^\sigma \|_{L^2} \to S \quad \text{uniformly}$$

Hence, we can assume by choosing R large enough that

$$\sup_{\sigma, t} \| v_+^\sigma \|_{L^2} < S < 2^{2/n} S$$

(iii) Let $R>0$ be as in (ii) and let
$R_2 = 4R = R_1^{-1}$.

Let
$$M := \{ u \in W_0^{1,2}(\Omega) : \int_\Omega |u|^{2^*} dx = 1 \}$$

and
$$F(u) := \int_\Omega x \cdot |\nabla u|^2 \, dx$$
(center of mass)

Assume that we do not have a non-trivial solution to (7.1) on Ω

$\rightarrow$ Ξ sat. $(P.S)_\beta$ $\forall \beta \in (S, 2$

Deformation lemma + covering $\rightsquigarrow$ For any $\delta > 0$ there exist a deformation Φ with
$$\Phi(M_{S_1}, 1) \subset M_{S+\delta}$$

(iv) Since Ω is smooth, there exists a neighborhood U of Ω such that
$$\pi(x) := \{ y \in \bar\Omega : dist(x,y) = min! \}$$
is well-defined and smooth

(iv) There is an $\delta > 0$ such that

$$F(H_{s+\delta}) \subseteq \mathcal{U}$$

Proof: If not, $\exists\, u_k \in H_{s+\frac{1}{k}} : u_k \notin \mathcal{U}$.

But we have seen in Exercise ... that

then

$$|u_m|^{2^*} dx \to \delta_{x^0} \quad \&$$

$$|\nabla u_m|^2 dx \to S\, \delta_{x^{(0)}}$$

for some $x^{(0)} \in \overline{\Omega}$.

(v) Then

$$h(\sigma, t) := \pi\left(F\left(\Phi(v_\sigma^{\phi}, t)\right)\right)$$

is well defined with

$$h(\sigma, 1) = \pi\left(F\left(\phi(v_\sigma, 1)\right)\right) =: x_\sigma \in \overline{\Omega}$$

and

$$h(\sigma, 1) = \sigma \quad \forall \sigma \in \Sigma$$

$\rightsquigarrow$ h is a contraction of Σ in Ω ↯