Recap: We analyzed solutions to the problem
\[ F(u) = \int F(x, u(x), Du(x)) \, dx \]
\[ \rightarrow \min! \quad \text{(or} \rightarrow \max!) \]
within the class of functions \( u \in C^1(\Omega, \mathbb{R}^m) \)
with boundary values \( y \), i.e.
\[ u = y \quad \text{on} \, \partial \Omega \]
for a fixed \( y \in C^1(\bar{\Omega}, \mathbb{R}^m) \) which
we denote by \( H_y \).

* If \( F \in C^1 \) we found that \( u \) solves the
weak form of the Euler-Lagrange equation, i.e.
\[ \delta F(u; h) = \int \left( \left( F_{\frac{\partial}{\partial x_j} x_k} (x, u(x), Du(x)) \frac{\partial}{\partial x_j} h^k \right) \right. \]
\[ + \left. F_{\frac{\partial}{\partial x_k} x_j} (x, u(x), Du(x)) h^k \right) \, dx \]
\[ = 0 \quad \forall h \in C^\infty_c(\Omega, \mathbb{R}^m) \]
(Einstein summation convention)

* If \( F, u \in C^2 \), we have
\[ - \sum_j F_{\frac{\partial}{\partial x_j} x_k} (x, u, Du) + F_{\frac{\partial}{\partial x_k} x_j} (x, u, Du) = 0 \]
on \( \mathbb{R} \)
for \( \forall k = 1, \ldots, m \).
Examples:

(i) Mechanical Systems:

- $N$ particles with masses $m_i$
- $x_i(t)$ (position at time $t$)
- $v_i(t) = (x_i(t), y_i(t), z_i(t)) \in \mathbb{R}^3$, $i = 1, \ldots, N$
- $v_i = \dot{x}_i$ = velocity

$$\mathcal{T}(v) = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{x}_i^2 = \text{Kinetic energy}$$

$$\mathcal{U}(t, u) = \text{potential energy}$$

Lagrangian: \[ \mathcal{L}(u, \dot{u}) = \mathcal{T}(\dot{u}) - \mathcal{U}(t, u) \]

Action: \[ \mathcal{S}(u) = \int_{t_1}^{t_2} \mathcal{L}(u, \dot{u}) \, dt \]

The principle of least action:

The system is trying to minimize the action.

\[ \Rightarrow \text{Euler-Lagrange equation} (x=1)^N \]

$$- \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{v}_i} \right) - \nabla \cdot \mathcal{U}(u, \dot{u}) = 0$$

(We do not sum over $i$)

(\Rightarrow\text{Newton's law of motion})
(ii) For $q > 1$ one can show that the Euler-Lagrange equation associated with
\[
\mathcal{F}(u) := \int \left( \frac{1}{2} (Du)^2 + \frac{1}{q} \, |u|^q \right) \, dx
\]
\[(x = \text{const})\]
is the partial differential equation
\[
A_{k}^{\partial} + \frac{2}{|u|^q - 2} \, u = 0 \text{ on } \mathbb{R}^2
\]

(iii) Minimal graphs: $m = 1,$
\[
A(u) := \int \sqrt{1 + |Du|^2} \, dx
\]
\[= \text{area of the graph of } u\]
Strong form of the E-L-equation:
\[
\frac{d}{dx} \left( \frac{\mathbf{d} u}{\sqrt{1 + |\mathbf{D} u|^2}} \right) = 0 \text{ in } \mathbb{R}^2;
\]

Surprisingly we can show the following:

Theorem 1.4: If $u \in H^m \cap C^2$ solves the minimal graph equation with boundary data $g$, then $u$ is the unique minimizer of the area in $H^m \cap C^2$.
Proof: We use the convexity of $F$. Let $v \neq u \in K \cap C^2$. Then

$$\mathcal{E}(v) - \mathcal{E}(u) = \int_{\mathbb{R}} \left( F(\nabla v) - F(\nabla u) \right) dx$$

Taylor expansion:

$$= \int_{\mathbb{R}} \left( \frac{\partial_i (v-u) \cdot F'(\nabla u)}{\bar{\nu}_i} \right) + \int_{\mathbb{R}} \left( \frac{1}{\bar{p}_i} \right) \frac{\partial_i (v-u) \cdot F'(\nabla u)}{\bar{\nu}_i} \right)$$

(Enter Lagrange) o

$$\cdot \partial_i (u-v)$$

$$dt$$

$> 0$, if we can show that the matrix

$$\left( a_{ij} \right)_{i,j=1,\ldots,n} = \frac{F(p;\nu)}{\bar{p}_i \bar{p}_j}$$

has only positive eigenvalues and $\nabla (u-v) \neq 0$. The last point holds, as $u \neq v$ but $u = v$ on $\partial \Omega$.

$$a_{ij} = \frac{F(p;\nu)}{\bar{p}_i \bar{p}_j} = \frac{\partial_i \left( \frac{\bar{p}_j}{\sqrt{1 + \bar{p}_j^2}} \right)}{\partial \bar{p}_i}$$

$$= \frac{S_{ij}}{\sqrt{1 + \bar{p}_j^2}} - \frac{R_{ij} \bar{p}_j}{(\sqrt{1 + \bar{p}_j^2})^3}$$
and hence \((a_{ij})\) has the Eigenvector \(p\) with Eigenvalue

\[
\frac{A}{\sqrt{1+p^2}} - \frac{p^2}{(\sqrt{1+p^2})^3} = \frac{A}{(\sqrt{1+p^2})^3}
\]

and the Eigenvector \(\frac{1}{\sqrt{1+p^2}}\) with multiplicity \((n-1)\) if \(p \neq 0\).

(Otherwise of course \((a_{ij}) = \delta_{ij}\))

Thus the proof is complete. \(\square\)
(iv) Minimal surfaces of revolution

Rotate
\[ u : [a, b] \to \mathbb{R}^+ \]
around the \( z \) axis \( \psi \)

\[
\text{area} = 2\pi \int_a^b u \sqrt{1 + u'^2} \, dx
\]

\[
= \int_a^b F(u, u') \, dx
\]

with \( \frac{d}{dx} F(g, p) = p \sqrt{1 + p^2} \)

Euler-Lagrange equation

\[
\frac{d}{dx} \left( u(x) \frac{u''(x)}{\sqrt{1 + (u')^2}} \right) = \sqrt{1 + (u')^2}
\]

(\( \Rightarrow \))

To solve this ODE we will use the following special case of Noether's theorem.

Theorem 1.5: Let \( F(y, p) \in C^2(\mathbb{R}^n \times \mathbb{R}^m) \) and \( u \in C^2([a, b], \mathbb{R}^m) \) satisfy the Euler-Lagrange equation \((x = \xi \in \mathbb{R})\)

\[
\frac{d}{dx} \left( \frac{F_{p_k}}{F_{y_k}}(u, u', u'') \right) + \frac{F_y}{F_{y_k}}(u, u', u'') = 0
\]
Then the quantity

\[ \phi(x) := (u^k) F_{px}(u, u^l) - F(u, u^l) \]

is constant \( \Box \)

\[ \text{Proof:} \quad \text{We have} \]

\[ \frac{d}{dx} \phi(x) = (u^k)^{\prime} F_{px}(u, u^l) \]

\[ + (u^k)^{\prime} \frac{d}{dx} \left( F_{px}(u, u^l) \right) - F_{px}(u, u^l) (u^k)^{\prime} \]

\[ \text{EL-equation} = 0 \]

\[ - F_{px}(u, u^l) (u^k)^{\prime} \]

\[ = 0 \quad \Box \]

**Remark:** (i) For mechanical systems
we have

\[ \phi(x) = \sum_{i=1}^{K} m_i v_i^2 - \frac{1}{2} \sum_{i=1}^{K} m_i \dot{v}_i^2 + U(x, t) \]

\[ = \bar{T} + U = \text{total energy of the system} \]

So Thm. 1.5 states the conservation of energy \( \Box \)
Back to minimal surfaces of revolution:

We get using Noether's theorem

$$u \cdot u' \frac{u'}{\sqrt{1 + (u')^2}} - u \sqrt{1 + (u')^2} = -c$$

$$= u \cdot \frac{-1}{\sqrt{1 + (u')^2}}$$

So

$$\frac{u}{\sqrt{1 + (u')^2}} = c$$

If we assume that \( u' > 0 \) we hence get

$$u' = \sqrt{\frac{u^2}{c^2} - 1}$$

which is a O.D.E. with separated variables

$$\int_{u_0}^{u} \frac{du}{\sqrt{\frac{u^2}{c^2} - 1}} = \int_{x_0}^{x} \frac{dx}{x}$$

$$= x - x_0$$

$$c \cdot (\text{arcosh} \left( \frac{u}{c} \right) - \text{arcos} \left( \frac{u_0}{c} \right)) = x - x_0$$

So a general solution has the form

$$u = c \cdot \text{cosh} \left( \frac{x - x_0}{c} \right)$$
The surface is called Catenoid and the profile curve is called chain line.

**Caution!**

(i) There is another minimal surface of revolution, the plane. But the profile curve then is not the graph of a function \( y = f(x) \).

(ii) All solutions of the associated E-L equations are all minimal surfaces (even if they do not minimize area).

\( \Rightarrow \) The later are called minimizing minimal surfaces.

(iii) If the circles are too far away from each other, there is no catenoid connecting them. \( \checkmark \)
Constraints

Three types of constraints (we will only discuss the last one)

Holonomic constraints: \( \mathcal{g}(x, u(x)) = 0 \), \( \forall x \in \mathbb{S}^2 \)

Examples:

(i) Minimize the Dirichlet energy

\[
\mathcal{D}(u) = \frac{1}{2} \int |Du|^2 dx
\]

within the class of functions satisfying

\[ |u|^2 = 1 \quad \text{in} \quad \mathbb{S}^2 \]

\( \Rightarrow \) harmonic maps

(or \( u(x) \in M \) where \( M \) is a submanifold of \( \mathbb{R}^n \))

(ii) typical for physical systems

Non-holonomic constraints:

\[ \mathcal{g}(x, u(x), Du(x)) = 0 \]
We will only deal with

Isoperimetric constraints:

\[ f(u) := \int \ G(x, u(x), Du(x)) \, dx = B \]

\( (u = c) \)

Theorem 1.6 (Lagrange multiplier)

Let \( F, G \in C^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}) \)

and \( c \in \mathbb{R} \) be given and let \( u \in H^1_0 \)

be a local minimize of \( f \) on \( H^1_0 \), i.e., let there be a \( \delta > 0 \)

such that

\[ f(u) \leq f(v) \quad \forall v \in H^1_0 \] with

\[ \| u - v \|_{H^1} \leq \delta, \]

where

\[ H^1_0 := \left\{ v \in H^1 : f(v) := \int G(x, u(x), Du(x)) \, dx \right\} \]

\[ = c \]

Furthermore, let there be a \( \eta \in C^\infty(\mathbb{R}, \mathbb{R}^m) \)

such that \( \delta \cdot \gamma (u; \eta) \neq 0 \).

Then there is a Lagrange multiplier \( \lambda \in \mathbb{R} \) such that

\[ \delta \gamma (u; \lambda) + \lambda \delta \gamma (u; \eta) = 0 \]
i.e. \( u \) is a critical point of \( \tilde{g} + \tilde{T} \varphi \).

**Proof:** Idea: reduce to two dimensions.

**Step 1:** For \( h \) and \( \varphi \) as in the theorem we look at

\[
g : \mathbb{C}^2, \mathbb{R}^2 \rightarrow \mathbb{R} \\
g(e, \tau) := g(u + \tau, h + \tilde{T} \varphi).
\]

Due to Thm. 1.2 the partial derivatives exist and we have

\[
\frac{dg(e, \tau)}{d \tau} = \int \left( \sum_{p,j} \gamma_k (x; u, \tau, \varepsilon) \frac{D_k e}{e} \right) \\
\cdot \frac{\partial g}{\partial \tau} + \int \left( \sum_{p,j} \gamma_k (x; u, \tau, \varepsilon) \frac{D_k e}{e} \right) \\
\cdot \frac{\partial e}{\partial \tau} \, dx
\]

which defines a continuous function (by the lemma of Lebesgue). Since for \( \frac{dg}{d \varepsilon} \) hence \( g \) is \( C^1 \).
Step 2: Since \( g(0,0) = c \) and
\[
\partial_2 g(0,0) = \delta g'(u, v) \neq 0
\]
the implicit function theorem gives us a function
\[
\epsilon : \mathbb{C} \rightarrow \mathbb{R}
\]
(for a \( \tau_0 > 0 \) small enough) with
\[
\epsilon(0) = 0
\]
and
\[
g(u + \tau h + \epsilon(\tau) v) = g'(\epsilon(\tau), \tau) = 0.
\]
Hence, \( u + \tau h + \epsilon(\tau) v \in M \) for all \( \tau \in \mathbb{C} \). Thus,

Step 3: We have
\[
0 = \frac{\partial}{\partial \tau} g(u + \epsilon(\tau), \tau)
\]
\[
= \delta g'(u, v) + \epsilon'(\tau) \delta g(u, v) + \epsilon(\tau) \delta g'(u, v)
\]
so
\[
\epsilon'(\tau) = -\frac{\delta g'(u, v)}{\delta g(u, v) \neq 0}.
\]
Examples:

(i) **Chain line**: Chain hanging between two points

Assumption:

\( \text{line} = \text{graph of} \quad y = \text{graph of} \quad u : \mathbb{R} \to \mathbb{R} \)

\( \Rightarrow \) We have to minimize the potential energy

\[
E_{\text{pot}}(u) = \int_{a}^{b} u(x) \sqrt{1 + (u'(x))^2} \, dx
\]

under the constraint

\[
L(u) = \int_{a}^{b} \sqrt{1 + (u'(x))^2} \, dx = \text{const.}
\]

Since

\[
SL(u, h) = \int_{a}^{b} \frac{u'(x)}{\sqrt{1 + (u'(x))^2}} h(x) \, dx
\]

there is an \( \mathbf{x} \in C_{c}^{\infty}(a, b) \) such that

\[
SL(u, h) \neq 0
\]

unless \( u \) is constant.
So if $h$ is a non-constant minimizer it satisfies

$$
\delta \int_{\text{pot}} (u; h) + \int \delta L (u; h) = 0
$$

$\forall h \in C^0_c ((a, b))$

The strong form of this equation is

$$
- \frac{d}{dx} \left( \frac{(u+1) u'}{\sqrt{1+(u')^2}} \right) + \sqrt{1+(u')^2} = 0
$$

If we set $\tilde{u} = u + \epsilon$ we get

$$
- \frac{d}{dx} \left( \tilde{u} - \frac{\tilde{u}'}{\sqrt{1+(\tilde{u}')^2}} \right) + \sqrt{1+(\tilde{u}')^2} = 0
$$

which is the Euler-Lagrange equation for minimal surfaces of revolutions

$$
\tilde{u}(x) = c \cdot \cosh \left( \frac{x-x_0}{a} \right)
$$

for a suitable $c > 0$. 
(ii) Isoperimetric Problem for graphs:

Problem: \( m = 1 \)

\[
\text{Vol}(v) = \int_0^1 \sqrt{v(x)} \, dx \to \max \quad \text{under the constraint}
\]

\[
A(v) = \int_0^1 \sqrt{1 + |v'|^2} \, dx = \text{const}
\]

(with boundary data \( y \in C^1(\overline{\mathbb{R}}, \mathbb{R}) \))

for \( h \in C^\infty_c(\mathbb{R}) \) we have

\[
\Delta \text{Vol}(v; h) = \int_0^1 h(x) \, dx
\]

and

\[
\Delta A(v; h) = \int_0^1 \frac{\nabla v \cdot \nabla h}{\sqrt{1 + |\nabla v|^2}} \, dx
\]

\[
\begin{align*}
\mathcal{F}^2 &= -\int_0^1 \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \, h \, dx \\
\text{if } u &\in C^2 \\
\end{align*}
\]

\[
= \hat{u}(x) = \text{mean curvature of the graph of } u.
\]
If $H_u(x) \neq 0$, i.e. if $u$ is not a minimal graph, we get the Euler-Lagrange equation

$$\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 2 \quad \text{in} \quad \Omega$$

for a suitable constant $2$.

$\Rightarrow \quad$ CMC - surface.

(constant mean curvature)

(To verify $u = 1$ one can get

$$u_{\bar{u}} = \left( \frac{u}{u(x)^2} \right) = \text{constant} = 2$$

curvature of the curve $x \rightarrow (x, u(x))$)

Hence, $(x, u(x))$ lies on a circle with radius $\frac{1}{\sqrt{2}}$. 