Example:

(i) Let $\Omega \subset \mathbb{R}^n$ be measurable, $p \in (1,\infty)$ and

$$\mathcal{F}: L^p(\Omega) \to \mathbb{R}$$

$$\mathcal{F}(u) = \int_\Omega |f|^p \, dx$$

Claim: $\mathcal{F}$ is C^1 on $L^p(\Omega)$.

Proof:

Step 1: $\boxed{\mathcal{F} \text{ is Gâteaux differentiable!}}$

For $u, h \in L^p(\Omega)$ we have

$$\frac{\mathcal{F}(u+\varepsilon h) - \mathcal{F}(u)}{\varepsilon} = \frac{1}{\varepsilon} \int_\Omega \left(|u+\varepsilon h|^p - |u|^p \right) dx$$

$$\underset{\text{fund. thm of calc}}{=} \frac{1}{\varepsilon} \int_\Omega \int_0^1 \frac{d}{d\tau} |u + \tau\varepsilon h|^p \, d\tau \, dx$$

$$= p \cdot \int_\Omega \int_0^1 \underbrace{|u+\tau\varepsilon h|^{p-2}(u+\tau\varepsilon h)\cdot h}_{\substack{\leq \frac{p-1}{p} |u+\tau\varepsilon h|^p + \frac{1}{p}|h|^p \\ \leq C_p (|u|^p + |h|^p) \in L^1}} \, dt \, dx$$

$a \cdot b \leq \frac{p-1}{p} a^{p/(p-1)} + \frac{1}{p} b^p$

$(a+b)^p \leq 2^{p-1}(a^p + b^p)$

$$\xrightarrow{\text{Lebesgue}} p \int_\Omega |u|^{p-2} u \cdot h \, dx \quad \text{as } \varepsilon \to 0.$$

$$\delta \mathcal{F}(f; g) \int \frac{|f(x)-f(y)|^{p-2}\langle f(x)-f(y), g(x)-g(y)\rangle}{|x-y|^{n+sp}} \, dx\, dy$$

$$f_n \to f \quad \text{in} \quad W^{s,p}$$

$$(a+b)^{\alpha} = 2^{\alpha}\left(\tfrac{a}{2}+\tfrac{b}{2}\right)^{\alpha} \geq 2^{\alpha-1}(a^{\alpha} + b^{\alpha})$$

wie (davon ...) gleichwas

$$\delta \mathcal{F}(f_n; g) - \delta \mathcal{F}(f; g)$$

$$= \int \frac{\left(\underline{|f_n(x)-f_n(y)|^{p-2}(f(x)-f(y))} - \dots + \right)}{|x-y|^{n+sp}} \, dx\, dy$$

$$w$$

$$\leq \int \frac{\left(\int \left(|f_n(x)-f_n(u)|^{p-2}(f(x)-f_n(y)) \dots \, du\right)^{\frac{p-1}{p}\frac{p}{p-1}}}{|w|^{n+sp}} \, dw$$

$$\leq \| \dots$$

$$\frac{\left(|f_n(x)-f_n(y)|^{p-2} f_n(x)-f_n(y) - |f(x)-f(y)|^{p-2} f(x)-f(y) \right)^{\frac{p}{p-1}}}{|w|^{n+sp}}$$

Since furthermore

$$\left(h \to p \int_\Omega |u|^{p-2} u \cdot h \, dx \right) \in (L^p(\Omega))'$$

$\tilde{\mathfrak{F}}$ is Gâteaux differentiable and

$$\tilde{\mathfrak{F}}'(u)\, h = p \int_\Omega |u|^{p-2} u \cdot h \, dx, \quad \forall u, h \in L^p(\Omega).$$

Step 2: $\boxed{\tilde{\mathfrak{F}}' : L^p(\Omega) \to (L^p(\Omega))' \text{ is continuous}}$

We will use the following convergence criterium:

Vitali's Theorem: Let (Ω, Σ, μ) and $f_n : \Omega \to \mathbb{R}$, $f : \Omega \to \mathbb{R}$ be in $L^1(\mu)$.

Then the following statements are equivalent:

(1) $f_n \to f$ in $L^1(\mu)$

(2) $f_n \to f$ in measure, i.e.

$$\mu(\{ |f_n - f| > \varepsilon \}) \to 0$$

for all $\varepsilon > 0$

and the f_n are uniformly integrable. //

Let $u_m \to u$ in $L^p(\Omega)$.
We will use Vitali's thm to prove that there is a subsequence m' with

$$|u_{m'}|^{p-2} u_{m'} \to |u|^{p-2} u$$
in $L^{p/p-1}(\Omega)$

This then implies
$$\mathcal{F}'(u_{m'}) \to \mathcal{F}'(u)$$
in $(L^p(\Omega))' = L^{p/p-1}(\Omega)$

We choose the subsequence such that

$u_m \to u$ pointwise almost everywhere.

(i) Hence,
$$|u_{m'}|^{p-2} u_{m'} - |u|^{p-2} u \to 0$$
pointwise almost everywhere. Hence,
$$\left| |u_{m'}|^{p-2} u_{m'} - |u|^{p-2} u \right|^{p/p-1} \to 0$$
in measure.

(ii) To see that $\left| |u_{m'}|^{p-2} u_{m'} - |u|^{p-2} u \right|^{p/p-1}$ is uniformly integrable, we estimate

$$\left| |u_{m'}|^{p-2} u_{m'} - |u|^{p-2} u \right|^{p/p-1}$$

(*) $$\leq C_p \left(|u_{m'}|^p + |u|^p \right)$$

$$\to C_p \left(|u|^p + |u|^p \right) \text{ in } L^1$$

Hence, by Vitali's theorem for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\mathcal{L}^n(E) \leq \delta \Rightarrow \int_E C_p \left(|u_{m'}|^p + |u|^p \right) dx$$

$$\leq \varepsilon$$

With (*) we get

$$\mathcal{L}^n(E) \leq \delta \Rightarrow \int_E \left(\left| |u_{m'}|^{p-2} u_{m'} - |u|^{p-2} u \right| \right)^{\frac{p}{p-1}} dx$$

$$\leq \varepsilon .$$

From (i) & (ii) & Vitali's thm we get

$$\int \left(\left| |u_{m'}|^{p-2} u_{m'} - |u|^{p-2} u \right| \right)^{p/p-1} dx$$

$$\to 0 \quad \text{as } m' \to \infty$$

This shows that

$$\mathcal{F}'(u_{m'}) \to \mathcal{F}'(u) \text{ in } \left(L^p(\Omega) \right)'.$$

We will now show that even

$$\mathcal{F}'(u_m) \to \mathcal{F}'(u) \quad \text{in } (L^p(\Omega))'$$

Assume that this was not true. Then there is a subsequence $u_{m'}$ with

$$(**) \quad \| \mathcal{F}'(u_{m'}) - \mathcal{F}(u) \|_{(L^p(\Omega))'} \geq \varepsilon$$

for some $\varepsilon > 0$. Since $u_{m'} \to u$ in $L^p(\Omega)$, we again get a subsequence $u_{m''}$ with

$$\| \mathcal{F}'(u_{m''}) - \mathcal{F}(u) \|_{(L^p(\Omega))'} \to 0$$

which contradicts $(**)$.

Hence, $\mathcal{F}' : L^p(\Omega) \to (L^p(\Omega))'$ is continuous

Step 3: $\mathcal{F}$ is Frechet-diff. in u

Proof: Let $\varepsilon > 0$. Since $\mathcal{F}'$ is cont. there is an $\delta > 0$ such that

$$\| \mathcal{F}'(v) - \mathcal{F}'(u) \|_{(L^p(\Omega))'} \leq \varepsilon$$

if $\quad \| u - v \|_{L^p(\Omega)} \leq \delta .$

Thus

$$\frac{|\mathcal{F}(u+v)-\mathcal{F}(u)-\mathcal{F}'(u)v|}{\|v\|_{L^p(\Omega)}}$$

$$= \frac{\left|\int_0^1 \mathcal{F}'(u+tv)v\,dt - \mathcal{F}'(u)v\right|}{\|v\|_{L^p}}$$

$$\leq \frac{\int_0^1 \overbrace{\|\mathcal{F}'(u+tv)-\mathcal{F}'(u)\|_{(L^p(\Omega))'}}^{\leq \varepsilon \text{ if } \|v\|\leq\delta} \|v\|_{L^p}}{\|v\|_{L^p}}$$

$$< \varepsilon \quad \text{if} \quad \|v\|_{L^p(\Omega)} \leq \delta$$

So $\mathcal{F}$ is Frechet-differentiable. $\square$

(ii) The Dirichlet energy

$$\mathcal{D} : W^{1,2}(\Omega) \to \mathbb{R}$$

$$\mathcal{D}(u) = \frac{1}{2} \int_{\Omega} |Du|^2 \, dx$$

and

$$\mathcal{F} : W^{1,p}(\Omega) \to \mathbb{R}$$

$$\mathcal{F}(u) := \frac{1}{p} \int_{\Omega} |Du|^p \, dx$$

for $p \in (1, \infty)$ are C^1.

Schauder estimates (~ reg. in Hölder spaces)

Theorem: Let L be an elliptic operator in non-divergence form with coefficients of class $C^{k,\alpha}$, $k \in \mathbb{N}_0$.

(i) (interior estimates) Suppose $u \in C^2(\Omega)$ s.t. $Lu = f \in C^{k,\alpha}(\overline{\Omega})$. Then $u \in C^{k+2,\alpha}(\Omega)$ and for any $\Omega' \subset\subset \Omega$ we have

$$\|u\|_{C^{k+2,\alpha}(\Omega')} \le C\Big(\|u\|_{L^\infty(\Omega)} + \|f\|_{C^{k,\alpha}(\overline{\Omega})}\Big).$$

(ii) (global estimates) If $\partial\Omega \in C^{k+2,\alpha}$ and $u \in C^0(\overline{\Omega})$ agrees with a function $u_0 \in C^{2+k,\alpha}(\overline{\Omega})$ on the boundary, then $u \in C^{2+k,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{2+k,\alpha}(\overline{\Omega})} \le C\Big(\|u\|_{L^\infty} + \|f\|_{C^{k,\alpha}(\overline{\Omega})} + \|u_0\|_{C^{2+k,\alpha}(\overline{\Omega})}\Big)$$

//

L^p- theory:

Same statement except that we exchange

$$C^{k+2,\alpha} \text{ by } W^{k+2,p}$$
$$C^{k,\alpha} \text{ by } W^{k,p}$$

and

$$\partial\Omega \in C^{k+2,\alpha} \text{ by } \partial\Omega \in C^{k+2,1}$$
$$a_{ij}, b_i, c \in C^{k,1}(\bar\Omega).$$

and assume $u \in W^{2,2}$ initially

Also important

Theorem of Friedrich:

Let $u \in W^{1,2}$ be a weak solution of

$$\begin{cases} -\partial_i(a_{ij}\partial_j u) + b_i\partial_i u + cu = f \text{ in } \Omega \\ u = \varphi \text{ on } \partial\Omega, \end{cases}$$

i.e.

$$\int_\Omega \left(a_{ij}\partial_j u\, \partial_i\phi + b_i\partial_i u\,\phi + cu\,\phi \right) = \int_\Omega f\phi\,dx$$

for all $\phi \in C_c^\infty(\Omega)$

and $u - \varphi \in W_0^{1,2}(\Omega)$

Furthermore,

Then $u \in W^{2,2}$ and

§ 3 Direct method

Most things will fit into the following abstract theorem

Theorem 3.1: Let M be a topological Hausdorff space and $\mathcal{F}: M \to \mathbb{R} \cup +\infty$ satisfy the condition of bounded compactness, i.e

$$\forall \alpha \in \mathbb{R} \text{ the set}$$

$$(3.1) \qquad K_\alpha := \underbrace{\{ u \in M : E(u) \leq \alpha \}}_{\text{(sublevel set)}}$$

is compact (in the sense of Heine Borel)

Then $\mathcal{F}$ is bounded from below and attains its infimum.
The same holds true if instead of (3.1) we assume that K_α is sequentially compact. //

Proof: We assume $\mathcal{F} \not\equiv \infty$, set $\alpha_0 = \inf_M \mathcal{F} \geq -\infty$ and choose a strictly decreasing subsequence $\alpha_m \downarrow \alpha_0$ as $m \to \infty$

Then the sets $K_m := K_{\alpha_m}$ are non-empty compact sets satisfying

$$K_{m+1} \subseteq K_m .$$

Hence, $\bigcap_{m \in \mathbb{N}} K_m \neq \emptyset$, i.e. there is

a $u \in M$ with $u \in K_m \ \forall m \in \mathbb{N}$.
Thus $\tilde{\mathcal{F}}(u) \leq \alpha_m \ \forall m \in \mathbb{N}$.
We calculate

$$\tilde{\mathcal{F}}(u) \leq \lim_{m \to \infty} \alpha_m = \alpha_0$$

$$= \inf_M \tilde{\mathcal{F}} \leq \tilde{\mathcal{F}}(u)$$

Thus $-\infty < \tilde{\mathcal{F}}(u) = \inf_M \tilde{\mathcal{F}}$.

(ii) For the case that the K_α are sequentially compact, we chose a minimizing sequence $(u_m) \subseteq M$, i.e. a sequence satisfying

$$\lim_{m \to \infty} \tilde{\mathcal{F}}(u_m) = \inf_M \tilde{\mathcal{F}} .$$

Fixing $\tilde{\alpha} > \alpha_0 = \inf_M \tilde{\mathcal{F}}$ we get that

$$u_m \in K_{\tilde{\alpha}} \quad \text{for almost all } m \in \mathbb{N}$$

K_α seqn. compact $\Rightarrow$ $u_m \to u$ for a $u \in M$.

Since for all $\alpha > \alpha_0$ we have

$u_m \in K_\alpha$ for m large enough

and K_α are closed we get

$u \in K_\alpha \quad \forall \alpha > \alpha_0 = \inf_M \tilde{\mathcal{F}}$

i.e. $\tilde{\mathcal{F}}(u) \leq \alpha$.

Hence, $\tilde{\mathcal{F}}(u) \leq \inf_M \tilde{\mathcal{F}} \leq \tilde{\mathcal{F}}(u)$

which leads to

$$-\infty < \tilde{\mathcal{F}}(u) = \inf_M \tilde{\mathcal{F}} \qquad \square$$

Remark:

(i) (3.1) implies that

$$\{ u \in H : \tilde{F}(u) > \alpha \} = H \setminus K_\alpha$$

is open, i.e. $\tilde{F}$ is lower semicontinuous

Lower semicont. is easier to satisfy the more open sets we have while compactness is easier if we have fewer sets. To satisfy both one has to find the right topology (in our cases usually the weak top. on some Sobolev space). but there are problems where this is not possible

Remark:

(i) Under (3.1) we get

$$\{ u \in M ; E(u) > \alpha \} = M \setminus K_\alpha \text{ is open}$$

=> E is lower semi-continuous (or sequentially lower semicont)

(ii) Lower semicont. easier to satisfy the more open sets we have (the coarser the topology is)

Compactness: less open sets make things easier

~> One has to addept the topology to the problem
In most of our case, some Sobolevspace will work.

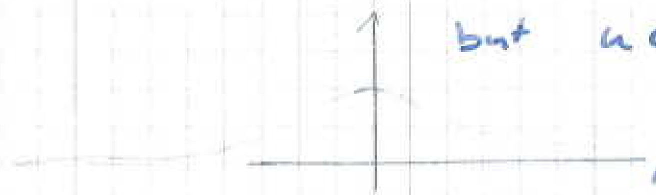
Much more useful for us is the following consequence:

Theorem 3.2: Let $M \subseteq X$ be a weakly closed subset of a reflexive Banach space X and

$$\mathcal{F} : M \to \mathbb{R} \cup +\infty$$

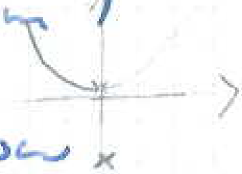
be coercive and sequentially weakly lower semicont., i.e (s.w.l.s)

(1) $\mathcal{F}(u) \to \infty$ as $\|u\|_X \to \infty$ but $u \in M$.



(2) If $u_m \overset{w}{\rightharpoonup} u$ weakly, then

$$\mathcal{F}(u) \le \liminf_{m \to \infty} \mathcal{F}(u_m)$$



Then $\mathcal{F}$ is bounded from below and attains its infimum in M //

Rem:

(1) excludes examples like
$\mathcal{F} : \mathbb{R} \to \mathbb{R}$, $\mathcal{F}(x) = \exp -x$

(2) excludes $\mathcal{F} : \mathbb{R} \to \mathbb{R}$, $\mathcal{F}(x) = |x|^2$
if $x \neq 0$ and $\mathcal{F}(0) = -1$ ∇

Proof: Let $(u_m) \subseteq M$ be a minimizing sequence, i.e. let

$$\lim_{m \to \infty} \mathcal{F}(u_m) = \inf_M \mathcal{F}.$$

Then $\mathcal{F}(u_m)$ is uniformly bounded and hence using the coerciveness

$$\sup_m \| u_m \|_X < \infty.$$

Eberlein $\Rightarrow$ After going to a subsequence
- Smulia we can assume that

$$u_m \to u \quad \text{weakly.}$$

As M is weakly closed, we get $u \in M$ and thus

$$\mathcal{F}(u) \overset{(s.w.l.s)}{\leq} \liminf_{m \to \infty} \mathcal{F}(u_m)$$

$$= \inf_M \mathcal{F} \leq \mathcal{F}(u).$$

So,

$$-\infty < \mathcal{F}(u) = \inf_M \mathcal{F}$$

□