§ 3.1 Examples

$\boxed{1}$ One can use the fact that $\|\cdot\|_X$ is sequentially weakly lower semi cont. for every reflexive Banach space.

Take $W_0^{1,p}(\Omega)$, $\Omega \subseteq \mathbb{R}^n$

equipped with the norm

$$\|u\|_{W_0^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}}$$

Due to Poincaré the norm is equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$ as

$$\|u\|^p_{W_0^{1,p}(\Omega)} \leq \|u\|^p_{W^{1,p}(\Omega)}$$

$$= \int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx$$

$$\underset{\text{Poincaré}}{\leq} (C+1) \int_\Omega |\nabla u|^p dx = (C+1) \cdot \|u\|_{W_0^{1,p}}$$

Hence,

$\|\cdot\|_{W_0^{1,p}(\Omega)}$ is s.w.l.s $\overset{\checkmark}{\circ}$

Theorem 3.3: Let $f \in \left(W_0^{1,2}\right)'$ and

$$\tilde{\mathcal{F}}: W_0^{1,2}(\Omega) \to \mathbb{R}$$

$$\tilde{\mathcal{F}}(u) := \frac{1}{2} \int |\nabla u|^2 \, dx + f(u).$$

Then $\inf\limits_{W_0^{1,2}} \tilde{\mathcal{F}}$ is attained by some function $u \in W_0^{1,2}$ which weakly solves

$$(3.1) \begin{cases} \Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof:

(i) Since

$$\tilde{\mathcal{F}}(u) = \frac{1}{2} \|u\|^2_{W_0^{1,2}} + f(u)$$

$$\geq \frac{1}{2} \|u\|^2_{W_0^{1,2}} - C \cdot \|f\|_{\left(W_0^{1,2}\right)^*} \|u\|_{W_0^{1,2}}$$

$$\underset{ab \leq \varepsilon a^2 + C_\varepsilon b^2}{\geq} \frac{1}{4} \|u\|_{W_0^{1,2}} - C \|f\|_{\left(W_0^{1,2}\right)^*}$$

$$\to \infty \quad \text{as} \quad \|u\|_{W_0^{1,2}} \to \infty$$

$\tilde{\mathcal{F}}$ is coercive.

(ii) Since for $u_m \overset{w}{\rightharpoonup} u$ we have

$$\liminf_{m \to \infty} \|u_m\|_{W_0^{1,2}} \geq \|u\|_{W_0^{1,2}}$$

and $f(u_m) \to f(u)$ (by the def. of weak conv)

we get $\liminf_{m\to\infty} \mathcal{F}(u_m) \geq \liminf_{m\to\infty} \mathcal{F}(u)$.

(iii) Applying Thm. 3.2 we get that there is a minimizer u which satisfies

$$0 = \frac{d}{dt} \frac{\mathcal{F}(u+th) - \mathcal{F}(u)}{t} \Big|_{t=0}$$

$$= \int \left(\nabla u \ \nabla h \right) dx + f(h)$$

$$\forall \, h \in W_0^{1,2}(\Omega)$$

i.e. u is a weak solution to

$$\begin{cases} + \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

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$\boxed{2}$ (Degenerate elliptic equations)

We can use the approach from $\boxed{1}$ also to prove

Theorem 3.4: For $\Omega \subset\subset \mathbb{R}^n$ and $p \in (1, \infty)$, $f \in (W^{1,p})'$

$$\mathcal{F} : W_0^{1,p}(\Omega) \to \mathbb{R}$$

$$\mathcal{F}(u) = \frac{1}{p} \int |\nabla u|^p \, dx + f(u).$$

Then the minimum of $\mathcal{F}$ is attained by u which weakly sat.

$$(3.2) \qquad \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f$$

in the sense that

$$-\int |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = f(\phi) \quad \forall \phi \in C_c^\infty(\Omega).$$

Furthermor $(*)$ has at most one weak solution $u \in W_0^{1,p}(\Omega)$ so especially the minimizer of $\mathcal{F}$ is unique. $\square$

(ii) For the uniqueness we will use that the p-Laplacian is monotone:

Let $u \neq v$ be two weak solutions of (3.2). Then

$$\int \left(|\nabla u|^{p-2} u - |\nabla v|^{p-2} v \right) \phi \, dx$$

$$= 0 \qquad \forall \phi \in C_c^{\infty}(\Omega).$$

Using that $C_c^{\infty}(\Omega)$ is dense in $W_0^{1,p}(\Omega)$ and

$$\left(\phi \to \int \left(|\nabla u|^{p-2} u - |\nabla v|^{p-2} v \right) \phi d \right)$$

$$\in \left(W^{1,p}(\Omega) \right)'$$

we get even

$$\int \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) \phi \, dx$$

$$= 0 \qquad \forall \phi \in W_0^{1,p}(\Omega).$$

(iii) For $\phi = u - v$ we get

$$0 = \int_{\Omega} \left(|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right) (u-v) \, dx$$

$$= \int_{\Omega} \int_0^1 \frac{\partial}{\partial t} g\left(\nabla v + t \nabla(u+v) \right) (u-v) t \, dx$$

$g(x) = |x|^{p-2} x$

$$= \int_\Omega \int_0^1 \partial_i g_j \left(\nabla v + t(\nabla u - \nabla v) \right) (u-v)_i \, (u-v)_j \, dx$$

> 0 ↯

as

$$\partial_i g_j (x) = |x|^{p-2} \delta_{ij} + (p-2) |x|^{p-4} x_i x_j$$

and hence

$$\partial_i g_j (x) \cdot \eta_i \eta_j = |x|^{p-2} |\eta|^2 + (p-2) |x|^{p-4} (x \cdot \eta)^2$$

> 0 unless $x = 0$ or $\eta = 0$.

$\boxed{3}$ (A semi-linear elliptic equation)

Let $\Omega \subset\subset \mathbb{R}^n$, $f \in C^\infty(\overline{\Omega})$,
$2 < p < \infty$ and consider
the energy

$$\mathcal{F}(u) := \int_\Omega \left(\frac{|\nabla u|^2}{2} + \frac{|u|^p}{p} - fu \right) dx$$

on $W^{1,2}_0(\Omega)$.

Note: We have the Sobolev embedding

$$W^{1,2}_0(\Omega) \hookrightarrow L^q(\Omega)$$

$$\forall \; q \leq \frac{2n}{n-2} =: 2^*$$

$\Rightarrow$ $\mathcal{F}$ not even finite on $W^{1,2}_0(\Omega)$
if $q > 2^*$ (q supercritical exponent)

The associated Euler-Lagrange equation is

$$(3.3)\quad \begin{cases} -\Delta u + |u|^{p-2} u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Thm 3.4: $\mathcal{F}$ attains its minimum in $W_0^{1,2}(\Omega)$. The minimizer u is the unique weak solution of (3.3). //

Proof: $\mathcal{F}$ is coercive, since

$$\mathcal{F}(u) \underset{\text{Cauchy-Schwartz}}{\geq} \frac{1}{2}\|u\|^2_{W_0^{1,2}} - \|f\|_{L^2}\|u\|_{L^2}$$

$$\underset{\text{Poincare}}{\geq} \frac{1}{2}\|u\|^2_{W_0^{1,2}} - C\|f\|_{L^2}\|u\|_{W_0^{1,2}}$$

$$\geq \frac{1}{4}\|u\|^2_{W_0^{1,2}} - C\|f\|_{L^2}$$

$$C\cdot a\cdot b \le \frac{a^2}{4} + 2C^2b^2$$

$$\to \infty \quad \text{as} \quad \|u\|^2_{W_0^{1,2}} \to \infty.$$

Let $u_m \in \overset{\circ}{W}{}^{1,2}(\Omega)$ be a minimizing sequence, i.e.

$$\lim_{m\to\infty} \tilde{\mathcal{F}}(u_m) = \inf_{\overset{\circ}{W}{}^{1,2}(\Omega)} \tilde{\mathcal{F}}(u).$$

Coercivity of $\tilde{\mathcal{F}}$ & Eberlein Smulian $\Rightarrow$ We can assume that

$$u_m \overset{w}{\rightharpoonup} u \quad \text{in } \overset{\circ}{W}{}^{1,2}(\Omega)$$

and

$$u_m \to u \quad \text{in } L^2(\Omega)$$

(Rellich!)

$$u_m \to u \quad \text{pointwise almost everywhere.}$$

By Fatou's lemma

$$\int_\Omega |u|^p dx \le \liminf_{n\to\infty} \int_\Omega |u_n|^p dx$$

and hence

$$\tilde{\mathcal{F}}(u) \le \liminf_{n\to\infty} \tilde{\mathcal{F}}(u_n) \le \tilde{\mathcal{F}}(u).$$

So u is the minimizer we were looking for.

$\square$

Is is u a classical solution of (3. 3)?

Theorem 3. 5: Every weak solution $u \in W_0^{1,2} \cap L^p$ to (3.) belongs to $L^\infty(\Omega)$, and solves (3.) pointwise almost everywhere. If $\partial\Omega \in C^\infty$ we even have $u \in C^\infty(\overline{\Omega})$.

Proof: (Sketch) We have

$$\int_\Omega \left(\nabla u \, \nabla \phi + |u|^{p-2} u \phi \right) dx = \int f u \, dx$$

for all $\phi \in W_0^{1,2}(\Omega) \cap L^p(\Omega)$. Inserting $\phi = (u-k)_+ \in W_0^{1,2}(\Omega) \cap L^p(\Omega)$ we get

$$0 = \int_{\{u > k\}} \left(|\nabla u|^2 + \left(\underbrace{|u|^{p-2} u}_{= |u|^{p-1}} - f \right) (u-k)_+ \right) dx .$$

$$\geq \int_{\{u > k\}} \left(|u|^{p-1} - \|f\|_{L^\infty} \right) dx$$

If $k^{p-1} = \|f\|_{L^\infty}$, we deduce that

$$u \leq k .$$

Similarly $u \geq -k$, i.e. $\|u\|_{L^\infty}^{p-1} \leq \|f\|_{L^\infty}$

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Higher local Regularity: Now u solves weakly

$$-\Delta u = \tilde{f} := f - |u|^{p-2}u \in L^2$$

Thm of Friedrichs (local form) $\Rightarrow$ $u \in W^{2,2}_{loc}$ and

$$-\Delta u + |u|^{p-2}u = f$$ pointwise almost everywhere

$\tilde{f} \in L^p \ \forall p \in (1,\infty)$ $\xrightarrow{L^p\text{-theory}}$ $u \in W^{2,p}_{loc}(\Omega) \ \forall p \in (1,\infty)$.

Using Morrey embeddings we get

$u \in C^\alpha(\Omega)$ for an $\alpha \in (0,1)$

$\rightsquigarrow$ u solves

$$-\Delta u = \tilde{f} \in C^\alpha$$

pointwise almost everywhere

$\xrightarrow{\text{Schauder estimates}}$ $u \in C^{2,\alpha}_{loc}$ $\rightsquigarrow$ $u \in C^{4,\alpha}_{loc}$

$\rightsquigarrow$ $u \in C^\infty_{loc}$

Global Regularity: Same argument + $\overset{\circ}{\nabla}$
Just exchange local with global
spaces $\overset{\circ}{\nabla}$

4 (Signs matter a lot $\underset{\circ}{\nabla}$)

We look at

$$\mathcal{F}(u) := \int_\Omega \left(\frac{|\nabla u|^2}{2} \boxed{-} \frac{|u|^p}{p} - fu \right) dx.$$

which even for $2 < p < 2^*$ is not bounded from below

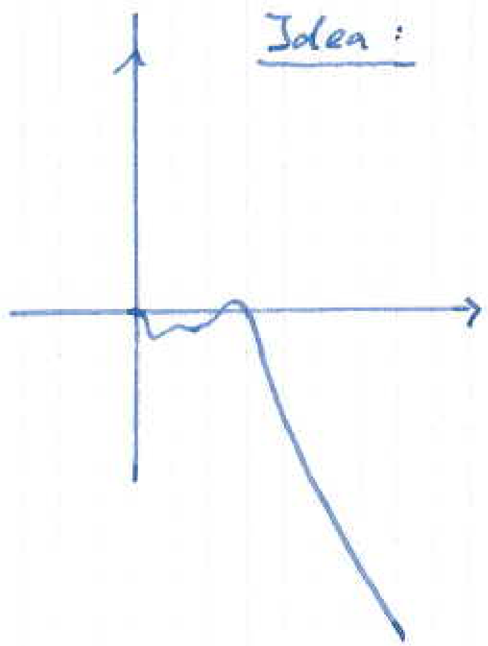
$$\mathcal{F}(\lambda u) = |\lambda|^2 \int_\Omega \frac{|\nabla u|^2}{2} dx$$

$$- |\lambda|^p \int \frac{|u|^p}{p} dx$$

$$+ \lambda \int fu \, dx$$

$$\to -\infty \quad \text{as } |\lambda| \to \infty$$

(unless $u \equiv 0$). $\rightsquigarrow$ no global min. $\underset{\circ}{\nabla}$



Idea: Should look like this, i.e. we should be able to find a local minimizer by looking at a suitable subset

$$M \subseteq W_0^{1,2}(\Omega) \; \underset{\circ}{\nabla}$$

Lemma (3.6): For $2 < p \le 2^*$ there exist $\varrho > 0$, $\mu > 0$ such that for $\|f\|_{L^2} \le \varrho$ we have

$$\inf_{\|\nabla u\|_{L^2} < \varrho} \mathcal{E}(u) < \inf_{\|\nabla u\|_{L^2} = \varrho} \mathcal{E}(u).$$

Proof: Since

$$\inf_{\|\nabla u\|_{L^2} < \varrho} E(u) \le E(0) = 0$$

and for $u \in W_0^{1,2}(\Omega)$ with $\|\nabla u\|_{L^2} = \varrho$ we have

$$E(u) \ge \frac{1}{2}\varrho^2 - c_1 \varrho^p - c_2 \varrho \|f\|_{L^2}$$

$$= \varrho \left(\frac{\varrho}{2} - c_1 \varrho^{p-1} - c_2 \|f\|_{L^2} \right)$$

Choose first ϱ so small that

$$\varrho/2 - c_1 \varrho^{p-1} \ge \varrho/4$$

and μ so small that $c_2 \mu \le \varrho/8$ to get

$$E(u) \ge \varrho^2/8. \quad \square$$

Theorem 3.7: Let $2 < p < 2^*$ and μ, ϱ be as in the last Lemma. Then for $f \in C^\infty(\overline{\Omega})$ with

$$\|f\|_{L^2} \leq \mu$$

there exists a $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$ with $\|\nabla u\|_{L^2} < \varrho$ such that

$$E(u) = \inf_{\|\nabla u\|_{L^2}} F(u).$$

u solves

$$-\Delta u - |u|^{p-2} u = f$$

weakly in Ω. //