3.2 Constraints

Using constraints we can show

Theorem 3. \( p \in (2, 2^*) \)

there is a solution \( u \) of

\[
\left\{ \begin{array}{ll}
Au = -|u|^{p-2}u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega \\
\end{array} \right.
\]

i.e. \( u \neq 0 \).

Proof: 1. We get this solution by minimizing

\[
\exists: \mathcal{M} \rightarrow \mathbb{R} \\
\exists(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 \, dx
\]

on the set

\[
\mathcal{M} := \left\{ u \in W^{1,2}(\Omega) : \int_{\Omega} |u|^{p^*} \, dx = 1 \right\}.
\]

We know from previous discussions, that \( \exists \) is coercive and (s. w. l. s. c.). Furthermore, Rellich-Kondrakov tells us that for \( u_n \rightharpoonup u \) weakly in \( W^{1,2}(\Omega) \), we have \( u_n \rightarrow u \) in \( L^p(\Omega) \) (note that \( p < 2^* \)).
and hence \( \int u^P \, dx = \lim_{\varepsilon \to 0} \int u^P \, dx = 1 \). So \( u \in K \) and thus we have shown that \( K \) is weakly closed.

**Thm 3.2** \( \tilde{u} \) is bounded from below and attains its minimum on \( K \), i.e., there is an \( u \in K \) with
\[
\tilde{F}(u) = \inf_{K} \tilde{F}.
\]

(i) As \( \tilde{F}, \tilde{g} \) are \( C^1 \) on \( W^{1,2}_0(\Omega) \) we get (as in the section about Lagrange multipliers) that there is a \( \lambda \in \mathbb{R} \) such that
\[
\tilde{F}'(u) - \lambda \tilde{g}'(u) = 0
\]
i.e.
\[
\int (\nabla u \cdot \nabla \lambda - \lambda \int u^{P-2} u \cdot h) \, dx = 0 \quad \forall h \in W^{1,2}_0(\Omega)
\]
i.e. \( u \) solves weakly
\[
+ A u + \lambda |u|^{P-2} u = 0 \quad \text{in} \ \Omega.
\]

(iii) Testing with \( u \) we get
\[
\int |\nabla u|^2 \, dx - \lambda = 0
\]
i.e.
\[
\lambda = \int |\nabla u|^2 \, dx > 0.
\]
Setting $\tilde{u} = 2^{1/2-p} u$ we get

$$\begin{cases}
-\Delta \tilde{u} = |u|^{p-2} u & \text{in } \mathcal{D} \\
\tilde{u} = 0 & \text{on } \partial \mathcal{D}
\end{cases}$$

The situation changes dramatically, if we assume that $p \neq 2^*$, i.e., for critical and supercritical exponents.

**Theorem 3.** Suppose $\mathcal{D} \subseteq \mathbb{R}^n$ is a smooth (possibly unbounded) domain which is strictly star-shaped with respect to the origin in $\mathbb{R}^n$ and let $\lambda \leq 0$. Then any solution $u \in W^{1,2}(\mathcal{D}) \cap C^2(\mathcal{D})$ of

$$\begin{cases}
-\Delta u = |u|^{p-2} u & \text{in } \mathcal{D} \\
u = 0 & \text{on } \partial \mathcal{D}
\end{cases}$$

vanishes identically. \(\therefore\)

The proof is based on the Pohožaev identity. \(\therefore\)
Lemma 3. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous and let \( u \in C^2(\Omega) \cap C^0(\mathbb{R}) \) be a solution to
\[
\begin{cases}
-\Delta u = f(u) & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial \Omega
\end{cases}
\]
in a domain \( \Omega \subset \mathbb{R}^n \). Then
\[
\frac{n-2}{2} \int_{\Omega} |\nabla u|^2 \, dx - \nu \int_{\partial \Omega} f(u) \, dx \\
+ \frac{1}{2} \int_{\partial \Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma = 0
\]
where \( \nu \) denotes the exterior unit normal.

Proof of Theorem 3. We get from Lemma
\[
\frac{n-2}{2} \int_{\Omega} |\nabla u|^2 \, dx - \nu \int_{\Omega} |u|^p \, dx \\
+ \frac{1}{2} \int_{\partial \Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma = 0.
\]
and testing within we get
\[
0 = \int_{\Omega} (-\Delta u u - |u|^p) \, dx \\
- \int_{\Omega} \left( |\nabla u|^2 - |u|^p \right) \, dx.
\]
hence

\[
\left( \frac{n-2}{2} - \frac{\nu}{p} \right) \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} \int_\Omega \left| \frac{\partial u}{\partial n} \right|^2 \, dx = 0.
\]

If now \( p > \frac{n-2}{2\nu} \), we get
\[
\int_\Omega |\nabla u|^2 \, dx = 0.
\]

If \( p = \frac{n-2}{2\nu} = 2^* \),

we can still deduce
\[
\int_\Omega \left| \frac{\partial u}{\partial n} \right|^2 \, dx = 0
\]

which implies \( u \equiv 0 \) by the principle of unique continuation.
Proof of Lemma:

We have

\[ 0 = \left( \Delta u + g(u) \right) (x \cdot \nabla u) \]

\[ = \text{div} \left( \nabla u (x \cdot \nabla u) \right) - |\nabla u|^2 \]

\[ - x \cdot \nabla \left( \frac{|\nabla u|^2}{2} \right) + x \cdot \nabla g(u) \]

\[ = \text{div} \left( \nabla u (x \cdot \nabla u) \right) \]

\[ - x \left( \frac{|\nabla u|^2}{2} + x \cdot \nabla g(u) \right) \]

\[ + \frac{n-2}{2} |\nabla u|^2 - n g(u) \]

Integrating over \( \Omega \) and using \( x \cdot \nabla u = x \cdot \partial_\nu \frac{\partial u}{\partial \nu} \) on \( \partial \Omega \) as \( u = 0 \) on \( \partial \Omega \), we get the lemma. \( \square \)
Remarks:

1. We will see later on that on topologically complicated domains and \( p = 2^* \) the critical exponent, there are non-trivial smooth solutions to

\[
\begin{align*}
-\Delta u &= |u|^{2^* - 2} u & \text{in } \mathbb{R}^N \\
\Delta u &= 0 & \text{on } \partial \Omega
\end{align*}
\]

(\textit{Dabrowsi \& Cazen})

for \( p > 2^* \) the analog to \( 2^* \) seems to be widely open (and much more complicated)
Plateau's Problem:

Let \( \Gamma \subset \mathbb{R}^3 \) be a smooth Jordan curve. Experiments convinced Plateau that every such curve is spanned by a surface of least area.

Model: Surface is that the topological type of a disc

\[
\mathcal{D} = \{ \mathbf{z} = (x, y) \in \mathbb{R} : x^2 + y^2 < 1 \}
\]

(naive) approach: Minimize area

\[
\mathcal{A}(u) = \int_{\mathcal{D}} \sqrt{\det(\nabla u^T \nabla u)} \, d\mathbf{z}
\]

\[
= \int_{\mathcal{D}} \sqrt{\left| \begin{array}{ll} u_x & u_y \\ u_y & u_x \end{array} \right|^2 - (u_x \cdot u_y)^2} \, d\mathbf{z}
\]

among "surfaces" \( u \in H^{1,2} \cap C^0(\overline{\mathcal{D}}; \mathbb{R}^3) \) satisfying the Plateau boundary condition

\[
u \big|_{\partial \mathcal{D}} : \partial \mathcal{D} \to \mathbb{R}^3 \quad \text{is a}
\]

monotone parametrization of \( \Gamma \) preserving the orientation.
Problem: A covariant under changes of the parametrizations.

Idea: [Douglas & Radó 1930]
There is a deep connection to the Dirichlet energy. If $u$ is conformal, i.e. $u \cdot u = 0$
$|u_x|^2 = |u_y|^2$ we have

$$H(u) = \int_{\Omega} \sqrt{|u_x|^2 |u_y|^2 - (u_x \cdot u_y)^2} \, d\Omega$$

$$= \frac{1}{2} \int_{\Omega} \left((|u_x|^2 + |u_y|^2) \right) \, d\Omega = D(u).$$

Actually we even have

Lemma: \text{inf } H(u) = \text{inf } D(u) \quad u \in C(\Omega) \quad u \in C(\Omega)
Plateau's problem is then reduced to

**Theorem 3.** For any $C^2$-embedded curve $\Gamma$ there exists a minimizer of the Dirichlet energy $D_{\Gamma}$.

To prove this theorem we still have to deal with conformal invariance of $D$, i.e., that

$$D(u) = D(u \circ g) \quad \forall g \in \mathcal{G}$$

where

$$\mathcal{G} = \left\{ g : \mathbb{R} \rightarrow \mathbb{R} | g(x) = e^{i\phi} \frac{\alpha x + \beta}{1 - \alpha x^2}, \quad \alpha \in \mathbb{C}, \quad |\alpha| \leq 1, \quad 0 \leq \phi < 2\pi \right\}$$

The conformal group of Möbius transformations of $\mathbb{R}$.

**Solution:** Use a three point condition. We use fixing the image of three points determines a unique $g \in \mathcal{G}$. 

\[ \text{\textsuperscript{\textdegree}} \]
So we fix a parametrization
\[ y: \mathbb{S}^1 \to \Pi \quad (e^{i \theta}) \]
and let
\[ \mathcal{E}(\Pi) = \left\{ u \in \mathcal{E}(\Pi) : \begin{array}{l}
u\left(e^{i \frac{2\pi k}{3}}\right) \\
u = y\left(e^{i \frac{2\pi k}{3}}\right)
\end{array}, \quad k = 1, 2, 3 \right\} \]

Now we are set to prove Theorem

Proof of Theorem:

Let \( u_m \in \mathcal{E}(\Pi) \) be a minimizing sequence of \( D \), i.e.
\[ \lim_{m \to \infty} D(u_m) = \inf_{\mathcal{E}(\Pi)} D. \]

First we will change this minimizing sequence to a nicer one. There is a unique \( f \in \mathcal{F} \) with
\[ f_k(e^{i \frac{2\pi k}{3}}) = f_{k-1} \circ y(e^{i \frac{2\pi k}{3}}) \]
where \( f_k = u_m |_{\mathbb{S}^1} \).
Then \( \tilde{u}_m = u_m \circ g_m \) is still a minimizing sequence and
\[
\tilde{u}_m \left( e^{2\pi i k/3} \right) = y \left( e^{2\pi i k/3} \right),
\]
i.e.
\[
\tilde{u}_m \in \mathcal{E}^*(\Pi).
\]

(ii) We can assume that
\[
\mathcal{E}(u_m) = \inf_{\forall \, \nu \in C^1 \cap C^0, \nu = u_m \text{ on } \partial \Omega} \mathcal{E}(\nu).
\]

If not, we exchange \( u_m \) by this minimizer, which belongs to \( C^\infty(\Omega) \cap C^0(\overline{\Omega}) \) and solves
\[
\Delta u_m = 0 \quad \text{in } \Omega.
\]

(iii) Claim: \( f_m = y u_m \mid_{\partial \Omega} : \Omega \to \mathbb{R}^3 \) is uniformly continuous.