The proof of this claim is based on

Lemma 3. (Courant - Lebesgue)

Let $f \in C^0(\bar{\Omega}, \mathbb{R}^3) \cap \mathcal{H}^1(\Omega, \mathbb{R}^3)$.
For every $z \in \bar{\Omega}$, $r \in (0,1)$ there is a $\rho \in (r^2, r)$ such that

$$\operatorname*{osc}_{\Omega \cap \partial B_\rho(z)} f \leq \left(\frac{4\pi}{\log \frac{1}{r}} \mathcal{D}(f) \right)^{1/2}$$

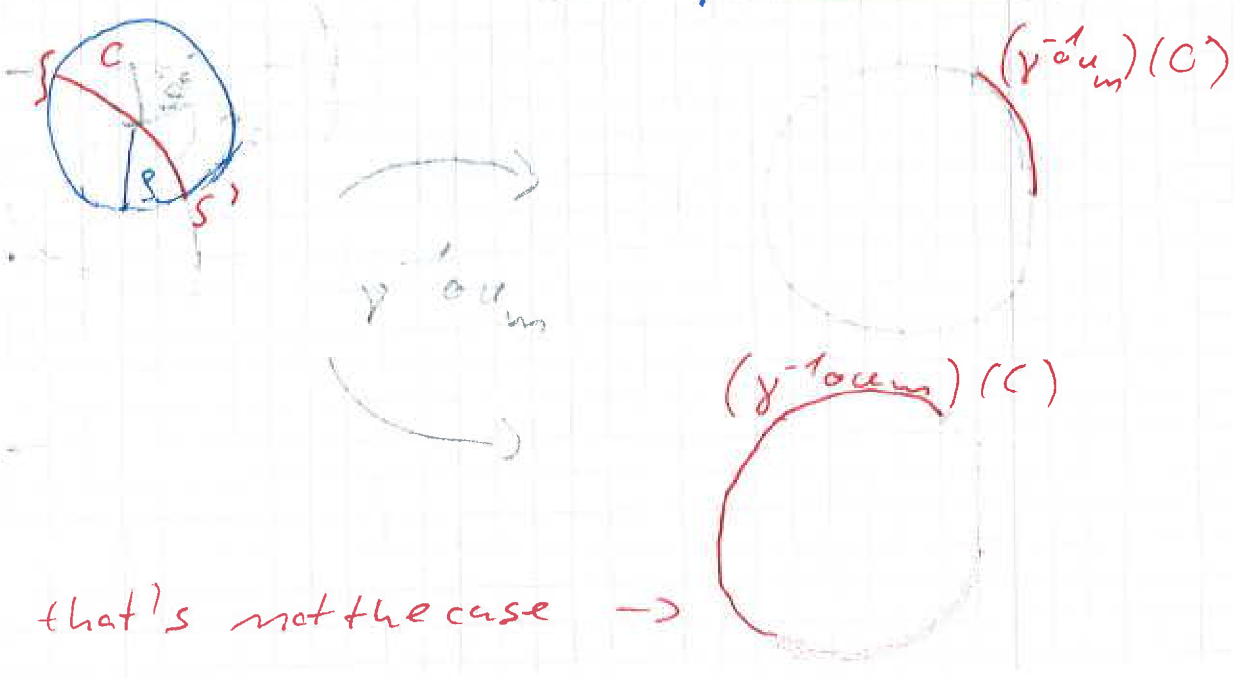
We will postpone the proof of this lemma and first use it to prove the claim:

Let $\varepsilon > 0$. Then the Courant-Lebesgue lemma tells us as $\mathcal{D}(u_m)$ is uniformly bounded and γ^{-1} is Lipschitz, that there is an $r_\varepsilon < \frac{\sqrt{2}}{2}$ such that for all $z \in \partial\Omega$ there is an $\rho \in (r_\varepsilon^2, r_\varepsilon)$ such that

$$\operatorname*{osc}_{\partial\Omega \cap \partial B_\rho(z)} (\gamma^{-1} \circ u_m) < \delta .$$

Let ζ, ζ' be the two points in $\partial \Omega \cap \partial B_{\varrho}(z)$. These points decompose the circle into two arcs and we denote the smaller arc by C. As $\varrho \le r_\varepsilon < \frac{\sqrt{3}}{2}$ this arc contains at most one of the points $e^{i 2\pi k/3}$, $k = 1, 2, 3$. Since $\gamma^{-1} \circ u_m$ is monotone, the image $(\gamma^{-1} \circ u_m)(C)$ is an arc connecting the two points $(\gamma^{-1} \circ u_m)(\zeta)$ and $(\gamma^{-1} \circ u_m)(\zeta)$. Since $e^{i 2\pi k/3}$, $k = 1, 2, 3$ are fix points of $\gamma^{-1} \circ u_m$, $(\gamma^{-1} \circ u_m)(C)$ also contains at most one of the points $e^{i 2\pi k/3}$ – and thus again must be the smaller arc.

two possibilities:



Thus we get

$$\operatorname{osc}_{\partial\Omega \cap B_{r_\varepsilon^2}(x)} (\gamma^{-1} \circ u_m)$$
$$\leq \operatorname{diam} \left(\gamma^{-1} \circ u_m \big|_{\partial\Omega \cap B_\delta(x_0)} \right)$$
$$\leq \varepsilon .$$

and $\gamma^{-1} \circ u_m|_{\partial\Omega}$ are uniformly cont. $\square$

$\overset{\gamma \in C^1}{\leadsto}$ $\quad u_m|_{\partial\Omega}$ is uniformly cont.

(iv) (iii) & Arzelà-Ascoli

$\Rightarrow$ After going to a subsequence

$$u_m \to u \quad \text{in} \quad C^0(\partial\Omega)$$

Since $\Delta u_m = 0$ in Ω the maximum principle tells us

$$\| u_m - u_\ell \|_{L^\infty(\Omega)}$$
$$\leq \| u_m - u_\ell \|_{L^\infty(\partial\Omega)} \leq \varepsilon$$

for ℓ, k large enough.

$\Rightarrow \quad u_m \to u \quad$ in $C^0(\Omega)$ for an u uniformly

Cauchy-estimates (or: elliptic regularity) $\Rightarrow$ All derivatives converge locally uniformly and hence for all $r > 1$:

$$\mathcal{D}(u|_{B_r(0)}) = \frac{1}{2} \int_{B_r(0)} |\nabla u|^2 dx$$

$$= \lim_{m \to \infty} \frac{1}{2} \int_{B_r(0)} |\nabla u_m|^2 dx$$

and hence

$$\mathcal{D}(u) \leq \lim_{m \to \infty} \int_{B_r(0)} |\nabla u_m|^2 dx$$

$$= \inf_{C(\Omega)} \mathcal{D} .$$

$\square$

Proof of the Courant-Lebesgue Lemma:

For $z, z' \in C_\varrho$ we have

$$|u(z) - u(z')| \le \int_{C_\varrho} \left|\frac{\partial f}{\partial s}\right| ds$$

$$\le \left(2\pi\varrho \int_{C_\varrho} \left|\frac{\partial f}{\partial s}\right|^2 ds\right)^{\frac{1}{2}}$$

and

$$\int_\Omega |\nabla u|^2 \, dz \ge \int_{r^2}^{r} \left(\varrho \left(\int_{C_\varrho} \left|\frac{\partial u}{\partial s}\right|^2 ds\right)\right)\frac{d\varrho}{\varrho}$$

$$\ge \left(\operatorname*{ess\,inf}_{r^2 \le \varrho \le r} \varrho \int_{C_\varrho} |\partial_s u|^2 ds\right) \cdot \int_{r^2}^{r} \frac{d\varrho}{\varrho}$$

$$= |\log r| \operatorname*{ess\,inf}_{r^2 \le \varrho \le r} \left(\varrho \int_{C_\varrho} |\partial_s u|^2 ds\right)$$

Hence, there is a $\varrho \in (r^2, r)$ such that

$$|u(z) - u(z')| \le$$

$$\le \left(2\pi\varrho \int \left|\frac{\partial f}{\partial s}\right|^2 ds\right)^{1/2}$$

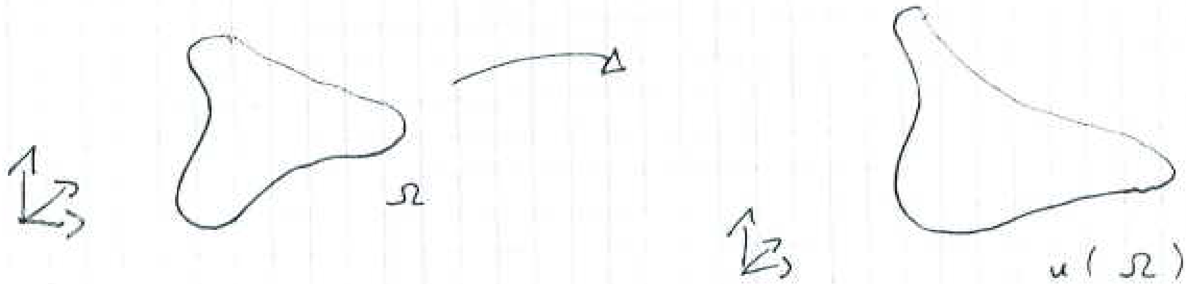
$$\le \left(2\pi \frac{\int_\Omega |\nabla u|^2 dz}{|\log \varrho|}\right)^{1/2} \qquad \square$$

§ 4 Compensated compactness:

Example 1: (Nonlinear elasticity)

$\Omega \subset\subset \mathbb{R}^3$, $u : \Omega \to \mathbb{R}^3$
"deformation" has the energy

$$\mathcal{F}(u) := \int_\Omega W(Du)\, dx$$



Natural conditions:

(1) $W(A) \to \infty$, if $\det(A) \to 0$ or ∞

(2) $W(AR) = W(A)$, $\forall R \in SO(3)$, $A \in L(\mathbb{R}^3)$
(isotropic material)

(3) $W(RA) = W(A)$, $\forall R \in SO(3)$, $A \in L(\mathbb{R}^3)$
(frame indifference)

Functions that satisfy (2) and (3) are for example

$$f(A) = \det A \,, \quad f(A) = |A|^2 = \sum_{i,j} |A_{ij}|^2.$$

Note that a function W sat. (1) cannot be convex !

Def 4.1 (Ball, 1977)

W is called poly-convex if

$$W(A) = g(A, \underbrace{A, \ldots, \det A}_{\text{minors of } A})$$

with g convex. //

Theorem 4.2 (Ball) Let $W: \mathbb{R}^{3\times 3} \to \mathbb{R}$ be continuous and poly-convex with

$$|A|^p \leq W(A) \leq C(1 + |A|^p$$

for some $p > 3$. Then $\mathcal{F}$ is (s.w.l.s) on $W^{1,p}(\Omega, \mathbb{R}^3)$. Hence, ~~for~~ for any $u_0 \in W^{1,p}(\Omega, \mathbb{R}^3)$ there is a $u \in u_0 + W_0^{1,p}(\Omega, \mathbb{R}^3)$ with

$$\mathcal{F}(u) = \inf_{v - u_0 \in W_0^{1,p}(\Omega, \mathbb{R}^3)} \mathcal{F}(v).$$

Proof: For simplicity we will assume that

$$W(A) = g(A, \det A)$$

for some convex continuous function g. Let $u_k \to u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^3)$. After passing to a subsequence if necessary, we may assume that $u_k \to u$ in $L^p(\Omega, \mathbb{R}^3)$ and

$$\mathcal{F}(u_k) \to \mathcal{F}_0 := \liminf_{k\to\infty} \mathcal{F}(u_k)$$

Lemma 4.3: $\det(Du_k) \to \det(Du)$ weakly in $L^{p/3}(\Omega)$ and in the sense of distributions, i.e.

$$\int_\Omega \det(Du_k)\,\varphi\,dx \Rightarrow \int_\Omega \det(Du)\,\varphi \qquad \forall \varphi \in C_c^\infty(\Omega).$$

//

From Lemma 4.3 we can conclude the proof of Theorem 4.2 as follows:

We have

$$(f_k) = \begin{pmatrix} Du_k \\ \det(Du_k) \end{pmatrix} \rightharpoonup \begin{pmatrix} Du \\ \det Du \end{pmatrix}$$

in $L^{p/3}(\Omega)$. By Banach-Alaoglu there are (a_l^k) with

$$0 \le a_l^k \le 1$$
$$\sum_{k \ge l} a_l^k = 1$$

and $\# \{ k : a_k^l \ne 0 \} < \infty$

such that

$$\sum_{k \ge l} \alpha_l^k \begin{pmatrix} Du_k \\ \det(Du_k) \end{pmatrix} \to \begin{pmatrix} Du \\ \det Du \end{pmatrix} = f$$

(strongly) in $L^{p/3}(\Omega)$. Since $g \ge 0$ and g continuous we get by Fatou's lemma

$$\mathcal{F}(u) = \int_\Omega g(Du, \det(Du))\,dx$$

$$\underset{\text{Fatou}}{\le} \liminf_{l \to \infty} \int_\Omega g\left(\sum_{k \ge l} \alpha_k^l \begin{pmatrix} Du_k \\ \det(Du_k) \end{pmatrix}\right) dx$$

$$\underset{g\ \text{conv}}{\le} \liminf_{l \to \infty} \sum_{k \ge l} \alpha_k^l \int_\Omega g(Du_k, \det(Du_k))\,dx$$

$$\le \limsup_{k \to \infty} \mathcal{F}(u_k) = \liminf_{k \to \infty} \mathcal{F}(u_k)$$
$$= \mathcal{F}_0. \qquad \square$$

We will prove the following stronger version of Lemma 4.3:

Lemma 4.4: Assume $n < p < \infty$ and $u_k \to u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^n)$, $\Omega \subset\subset \mathbb{R}^n$. Then

$$\det A_m^{(k)} \to \det A_m \quad \text{weakly in } L^{p/m}$$

where

$$A_m = \begin{pmatrix} \partial_1 u_1 & \cdots & \partial_1 u_n \\ \hline \\ \hline \partial_n u_1 & \cdots & \partial_n u_n \end{pmatrix}$$

← $(n-m)$ - rows & collums deleted ∇.

Proof: By induction on m, where statement for $m=1$ trivially holds. (We will only show the last induction step - all other steps are similar.)

(i) For smooth $w \in C^\infty(\Omega, \mathbb{R}^n)$ we get

$$\det(Dw) = \sum_{i=1}^{n} (\partial_{x_i} w_j)(\operatorname{cof} Dw)_{ij}$$

develop with respect to i-th line (Laplace)

$$\operatorname{cof} A = \det\begin{pmatrix} a_{11} & & \\ & a_{ij} & \\ & & a_{nn} \end{pmatrix}(-1)^{i+j}$$

delete i-th row & j-th collum

$$= \sum_{i=1}^{n} \partial_{x_i} \left(w_j \, (\mathrm{cof}\, Dw)_{ij} \right)$$

$$- w_j \sum_{i=1}^{n} \partial_{x_i} (\mathrm{cof}\, Dw)_{ij} .$$

(ii) From linear algebra we know

$$A^{-1} = \frac{1}{\det(A)} (\mathrm{cof}\, A)^T ,$$

i.e.

$$(*) \qquad \det(A)\, \delta_{ij} = \sum_{k=1}^{n} a_{ik} \, (\mathrm{cof}\, A)_{jk} .$$

Especially, for $i=j$

$$\frac{\partial \det(A)}{\partial a_{ij}} = (\mathrm{cof}\, A)_{ij} .$$

For $A = Dw$ we get differentiating $(*)$ in direction x_j

$$\sum_{j,l,k} (\mathrm{cof}\, Dw)_{kl} \quad \partial_{jk} w_l \quad \delta_{ij}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \partial_{ij} w_k \, (\mathrm{cof}\, Dw)_{jk}$$

$$+ \sum_{j,k=1}^{n} \partial_i u_k \quad \partial_j \left((\mathrm{cof}\, Dw)_{jk} \right)$$

Since
$$\sum_{j,l,k} (\operatorname{cof} Dw)_{kl}\, \partial_{jk} w_l\, \delta_{ij}$$
$$= \sum_{l,k=1}^{n} (\operatorname{cof} Dw)_{kl}\, \partial_{ik} w_l\,,$$

we deduce
$$\sum_{k=1}^{n} \partial_i w_k \left(\sum_{j=1}^{n} \partial_j (\operatorname{cof} Dw)_{jk} \right)$$
$$= 0 \qquad \forall\, i = 1, \dots n.$$

If $\det(Dw) \neq 0$, we get
$$\sum_{j=1}^{n} \partial_j (\operatorname{cof} Dw)_{jk} = 0 \ \forall\, k = 1, \dots n.$$

If $\det(Dw) = 0$, we get
$$\sum_{j=1}^{n} \partial_j \operatorname{cof}(Dw + \varepsilon)_{jk} = 0 \quad \forall\, k = 1, \dots n$$

for small enough $\varepsilon > 0$, applying the argument to $w_\varepsilon = w + \varepsilon x$. and still get
$$\sum_{j=1}^{n} \partial_j \operatorname{cof}(Dw) = 0$$

letting $\varepsilon \to 0$.

Hence, the lines of $(\operatorname{cof} Du)$ are divergence free.

(iii) Combining (i) & (ii) we get

$$(**) \qquad \det(Dw) = \sum_{j=1}^{n} \partial_{x_j}\left(w_j \, (\operatorname{cof} Du)_{ij} \right)$$

and a standard approx. result shows that $(**)$ holds for all $w \in W^{1,p}$. So for $\varphi \in C_c^\infty(\Omega)$ we get

$$\int_\Omega \det(Du^{(k)})\, \phi \, dx = \sum_{j=1}^{n} \int \partial_{x_j}\left(u_j^{(k)} (\operatorname{cof} Du^{(k)})_{ij} \right) \cdot \phi \, dx$$

$$= - \sum_{j=1}^{n} \int u_j^{(k)} \, (\operatorname{cof} Du^{(k)})_{ij} \, \partial_{x_j} \phi \, dx$$

$$\xrightarrow[k\to\infty]{} - \sum_{j=1}^{n} \int u_j \, (\operatorname{cof} Du)_{ij} \, \partial_{x_j} \phi \, dx$$

$$= \int \det(Du) \, \phi \, dx$$

since by induction

$$(\operatorname{cof} Du^{(k)})_{ij} \rightharpoonup (\operatorname{cof} Du)$$

weakly in $L^{p/n-1}$

and

$$u_j^{(k)} \partial_{x_j} \phi \to u_j \partial_{x_j} \phi$$

strongly in $L^{p/n} = (L^{p/n-1})^*$

(iv) Since $\{u^{(k)}\}$ is bounded in $W^{1,p}(\Omega, \mathbb{R}^n)$ and $\det(Du_k) \le C\,|Du_k|^n$, we see that $\det(Du_k)$ is bounded in $L^{p/n}$. Hence any subsequence has a weakly converging subsequence in $L^{p/n}(\Omega)$ – which must converge to $\det Du$ by (iii). $\square$

Remark: The divergence structure of $\det(Du)$ can be easier seen using forms:

$$\det(Du)\, dx^1 \wedge \dots \wedge dx^n$$

$$= du^1 \wedge \dots \wedge du^n$$

$$\underset{dd=0}{=} d\left(u^1 (du^2 \wedge \dots \wedge du^n)\right)$$

Lemma 4.5 ("div-curl lemma", Murat – Tartar) $(n = 3)$

Let $u_k \xrightarrow{\omega} u$, $v_k \to v$ in $L^2(\Omega, \mathbb{R}^3)$ and suppose that

$$\operatorname{div}(u_k) = 0$$
$$\operatorname{curl}(v_k) = 0$$

weakly in Ω. Then

$$u_k \cdot v_k \to u \cdot v \quad \text{in } \mathcal{D}'$$

(i.e $\int u_k \cdot v_k \, \phi \to \int u v \, \phi \, dx$
$\forall \phi \in C_c^\infty(\Omega)$.)

Proof: For simplicity we will only consider periodic functions, i.e.

$$\Omega = \mathbb{R}^n / \mathbb{Z}^n = \mathbb{T}^n .$$

(i) Let $w_k \in W^{1,2}(\mathbb{T}^n)$ be the minimizers of

$$\int_{\mathbb{T}^n} \left(\frac{|\nabla w|^2}{2} + v_k \cdot \nabla w \right) dx$$

within $\{ w \in W^{1,2}(\mathbb{T}^n) :$

$$\int_{\mathbb{T}^n} w \, dx = 0 \}$$

Then w_k is a weak solution of

$$-\Delta w_k = \operatorname{div} v_k + \lambda$$

and testing with $\phi \equiv 1$ we get

$$\lambda \cdot \int_{\mathbb{T}^n} 1\,dx = 0$$

So

$$\boxed{-\Delta w_k = \operatorname{div} v_k} .$$

Furthermore,

$$\|\nabla w_k\|_{L^2}^2 = -\int v_k \nabla w_k \, dx$$

$$\leq \|v_k\|_{L^2} \|\nabla w_k\|_{L^2}$$

$$\rightsquigarrow \quad \|w_k\|_{W^{1,2}} \underset{\text{Poincaré}}{\leq} C \|\nabla w\|_{L^2}$$

$$\leq \sup_k C \|v_k\|_{L^2} < \infty .$$

(ii) Let $h_k = v_k + \nabla w_k$. Then

$$\operatorname{div} h_k = \operatorname{div} v_k + \Delta w_k = 0$$

and

$$\operatorname{curl} h_k = \operatorname{curl} v_k = 0 .$$

As

$$\Delta = \nabla \operatorname{div} - \operatorname{curl} \operatorname{curl}$$

we deduce that h_k is harmonic in $\mathbb{T}^n$ and hence a constant function ?

Then

$$\|\nabla v_k\|_{L^2}^2 = \int_{T_n} (-\Delta w_k \, w_k)\, dx$$

$$= - \int v_k \, \nabla w_k \, dx$$

$$\leq \|v_k\|_{L^2} \|\nabla v_k\|_{L^2}$$

i.e.

$$\|w\|_{W^{1,2}} \leq \| \nabla w_k \|_{L^2} \leq C \| v_k \|_{L^2} \leq C < \infty$$

(ii) Let $h_k = v_k + \nabla w_k$. Then

$$\operatorname{div} h_k = \operatorname{div} v_k + \Delta w_k = 0$$

$$\operatorname{curl} h_k = \operatorname{curl} v_k = 0$$

and hence h_k is harmonic, as

$$\Delta h_k = \nabla \operatorname{div} h_k - \operatorname{curl} \operatorname{curl} h_k = 0$$

in $\mathbb{R}^3$.

(iii) Let us assume that

$$w_k \overset{w}{\rightharpoonup} w \quad \text{in} \quad W^{1,2}$$

$$h_k \overset{w}{\rightharpoonup} h \quad \text{in} \quad W^{1,2}.$$

But h_k is constant so

$$h_k \to h \quad \text{in} \quad L^2(T^n)$$

(iii) Let us assume that

$$w_k \overset{\omega}{\rightharpoonup} w \quad \text{in} \quad W^{1,2}$$

$$h_k \overset{\omega}{\rightharpoonup} h \quad \text{in} \quad W^{1,2}$$

Since h_k is constant we get $h_k \to h$ in $L^2(T^n)$ and get

$$\int_{T^n} u_k \cdot v_k \, \varphi \, dx = \int_{T^n} u_k \, (h_k - \nabla w_k) \varphi \, dx$$

$$= \int \Big(\underset{\substack{\downarrow \\ u}}{u_k} \underset{\substack{\downarrow \\ h}}{h_k} \varphi + (\underset{\substack{\downarrow \\ u}}{u_k} \underset{\substack{\downarrow \\ w}}{w_k}) \cdot \nabla \varphi + \underbrace{(\operatorname{div} u_k)}_{\overset{\shortparallel}{0}} w_k \varphi \Big) dx$$

(in L^2)

$$\to \int \big(u h \varphi + u w \cdot \nabla \varphi \big) \, dx$$

$$= \int u \cdot v \, \varphi \, dx.$$

(iv) Since thus any subsequence of $u_k \cdot v_k$ has a weakly convergent subsequence in $L^2(T^n)$ which converges by (iii) to $u \cdot v$, we get that

$$u_k v_k \overset{\omega}{\rightharpoonup} u v \quad \text{in} \quad L^2(T^n)$$

$\square$

Remarks:

(i) For $u_k(x), v_k(x) = \sin kx$ (on say $[0,1]$)

we have
$$u_k, v_k \overset{w}{\rightharpoonup} 0 \quad \text{in } L^2$$
but
$$u_k \cdot v_k \to \frac{1}{2} \qquad \nabla_c.$$

(ii) Lemma 4.5 is still true if
$$u_k \overset{w}{\rightharpoonup} u \quad \text{in } L^p \qquad , p,q \in (1,\infty)$$
$$v_k \to v \quad \text{in } L^q \qquad \frac{1}{p} + \frac{1}{q} = 1$$

(iii) We know that (for $n=3$)
$$\det(Du) = \sum_{k=1}^{3} \partial_i u^1 \left(\mathrm{cof}\, Du\right)_{i1}$$
where
$$\operatorname{div} \begin{pmatrix} (\mathrm{cof}\, Du)_{11} \\ (\mathrm{cof}\, Du)_{21} \\ (\mathrm{cof}\, Du)_{31} \end{pmatrix} = 0$$

and
$$\operatorname{curl} \begin{pmatrix} \partial_1 u^1 \\ \partial_2 u^? \\ \partial_3 u^3 \end{pmatrix} = 0$$

in the weak sense. Hence, $\det(Du)$ has the structure of the div-curl lemma ∇_c