Let us prove the following abstract "concentration-compactness lemma":

Theorem I.4: (P. L. Lions)

Let (μ_k) be a seq. of prob. measures i.e. $\mu_k \geq 0$, $\int_{\mathbb{R}^n} d\mu_k = 1$ $\forall k \in \mathbb{N}$.

Then there is a subsequence such that one of the following holds

(i) "compactness": There exists a sequence $(x_k) \subset \mathbb{R}^n$ such that $\forall \varepsilon > 0$ $\exists R > 0$ $\forall k \in \mathbb{N}$

$$\int_{B_R(x_k)} d\mu_k \geq 1 - \varepsilon .$$

(ii) "Vanishing": $\forall R > 0$:

$$\lim_{k \to \infty} \left(\sup_{x \in \mathbb{R}^n} \int_{B_R(x)} d\mu_k \right) = 0$$

(iii) "Dichotomy": There is $0 < \lambda < 1$ such that for all $\varepsilon > 0$ there is $(x_k) \subset \mathbb{R}^n$ and $R > 0$ such that for all $R' > R$ measures

$$0 \leq \mu_k^1, \mu_k^2 \leq \mu_k^1 + \mu_k^2 \leq \mu_k$$

with

$$\operatorname{supp}(\mu_k^1) \subset B_R(x_k)$$

$$\operatorname{supp}(\mu_k^2) \subset \mathbb{R}^n \setminus B_{R'}(x_k)$$

and

$$\limsup_{k\to\infty} \left(\left| \int_{\mathbb{R}^n} d\mu_k^1 - \lambda \right| + \left| \int_{\mathbb{R}^n} d\mu_k^2 - (1-\lambda) \right| \right) \leq \varepsilon$$

Remark: In (iii) the sequence (x_k) may be chosen independent of $\varepsilon > 0$ as in (i);

Proof: Based on the concentration function of P. Lévy

$$Q_k(r) = \sup_{x\in\mathbb{R}^n} \int_{B_r(x)} d\mu_k, \quad \forall r > 0.$$

Then

$$0 \leq Q_k(r) \leq 1$$

is non decreasing

$\Rightarrow$ There is a subsequence converging almost everywhere to a monoton function Q with

$$Q(0) = 0$$

Let $\lambda = \lim_{r \to \infty} Q(r) \in [0, 1]$

We show that $\lambda = 1$, $\lambda = 0$, $0 < \lambda < 1$ correspond to the cases (i), (ii) and (iii) in the theorem.

(α) Clearly $\lambda = 0$ implies (i)

(β) Claim: $\lambda = 1$ implies (ii)

Proof: For $R_0 > 0$ with $Q(R_0) > \frac{1}{2}$ we choose $(x_k) \in \mathbb{R}^n$ such that

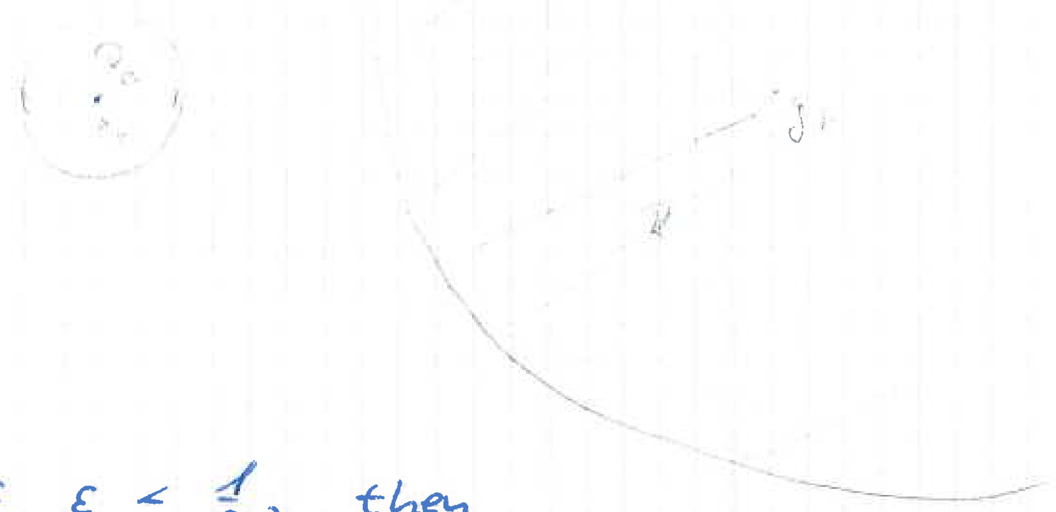
$$\int_{B_{R_0}(x_k)} d\mu_k \geq Q_k(R_0) - \frac{1}{k}.$$

Given $\varepsilon > 0$ we find $R > 0$ and $y_k \in \mathbb{R}^n$ with

$$Q(R) = \lim_{k \to \infty} Q_k(R) > 1 - \varepsilon$$

and

$$\int_{B_R(y_k)} d\mu_k \geq Q_k(R) - \frac{1}{k}, \quad k \in \mathbb{N}$$



If $\varepsilon < \frac{1}{2}$, then

$$B_{R_0}(x_k) \cap B_R(y_k) \neq \emptyset$$

as else

$$1 = \int_{\mathbb{R}^n} d\mu_k \geq \int_{B_{R_0}(x_k)} d\mu_k + \int_{B_R(y_k)} d\mu_k$$

$$\geq \frac{1}{2} + (1-\varepsilon) > 1 \quad \text{for } k \text{ large enough } \lightning$$

$$\rightsquigarrow \quad B_R(y_k) \subset B_{R_0+2R}(x_k), \ \forall k \geq k_0$$

and

$$\int_{B_{R_0+2R}} d\mu_k \geq \int_{B_R(y_k)} d\mu_k$$

$$\geq 1-\varepsilon, \quad \forall k \geq k_0$$

$\triangle$ claim!

(v) Claim: $0 < \lambda < 1 \Rightarrow$ (iii)

Proof: Given $\varepsilon > 0$, choose $R > 0$, $(x_k) \subset \mathbb{R}^n$ such that

$$Q(R) = \lim_{k \to \infty} Q_k(R) > \lambda - \varepsilon$$

$$\int_{B_R(x_k)} d\mu_k \geq Q_k(R) - \frac{1}{k} \geq \lambda - \varepsilon$$

for $k \geq k_0$. Let $R_k \to \infty$ be such that

$$Q_k(R_k) \leq Q(R_k) + \frac{1}{k} \leq \lambda + \varepsilon$$

$$\forall\, k \geq k_0$$

Let $R' > R$ and k_0 so large that $R_k > R'$.

Set

$$\mu_k^1 := \mu_k^1 \llcorner B_R(x_k)$$

$$\mu_k^2 := \mu_k^2 \llcorner (\mathbb{R}^n \setminus B_R(x_k))$$

Then for $k \geq k_0$ we have

$$-\varepsilon \leq \int_{B_R(x_k)} d\mu_k - \lambda = \int_{B_R(x_k)} d\mu_k^1 - \lambda$$

$$\leq \underbrace{\Bigg(\underbrace{\int_{B_{R_k}(x_k)} d\mu_k - 1}_{\substack{\| \\ \int_{\mathbb{R}^n} d\mu_k}} \Bigg) - (\lambda - 1)}_{= - \int_{\mathbb{R}^n - B_{R_k}(x_k)} d\mu_k = \int_{\mathbb{R}^n} d\mu_k^2}$$

$$\leq Q_k(R_k) - \lambda \leq \varepsilon$$

Hence,

$$\left(\left| \int d\mu_k^1 - \lambda \right| + \left| \int d\mu_k^2 - (1-\lambda) \right| \right)$$

$$\leq 2\varepsilon.$$

$\triangleleft$

(All other properties are obvious $\overset{!}{\nabla}$)

New observation in the critical case $p = 2^* = \frac{2n}{n-2}$ $(n \geq 3)$

Example 2: Fix $u \not\equiv u_1 \in C_c^\infty(B_1(0))$, $n \geq 3$. Scale (with

$$u_k(x) = k^{\frac{n-2}{2}} u_1(kx), \quad k \in \mathbb{N}$$

Then $u_k \in C_c^\infty(B_1(0))$, $u_k \to 0$ a.e. and

$$\|\nabla u_k\|_{L^2}^2 = \int |\nabla u_k|^2 \, dx$$

$$= k^{n-2+2} \int |\nabla u_1(kx)|^2 \, dx$$

$$\underset{\text{Subst. } y = kx}{=} \int_{B_1(0)} |\nabla u_1|^2 \, dx = \|u_1\|_{L^2}^2 =: A$$

and

$$\|u_k\|_{L^{2^*}}^{2^*} = k^n \int_{B_1(0)} |u_1(kx)|^{2^*} \, dx$$

$$= \int_{B_1(0)} |u_1(x)|^{2^*} \, dx =: B$$

i.e.

$$|\nabla u_k|^2 \, dx \xrightarrow{\omega^*} A \, \delta_{\{x=0\}}$$

$$|u_k|^{2^*} \, dx \quad B \, \delta_{\{x=0\}}.$$

In fact, much more can be said

Theorem (P.L. Lions):

Let $u_k \rightharpoonup u$ in $\mathring{H}^{1,2}(\mathbb{R}^n)$ and suppose that

$$\mu_k = |\nabla u_k|^2 dx \overset{w^*}{\rightharpoonup} \mu$$

$$\nu_k = |u_k|^{2^*} dx \overset{w^*}{\rightharpoonup} \nu$$

as $k \to \infty$.

Then there exists a number $J \in \mathbb{N}$ of concentration points $x^{(j)} \in \mathbb{R}^n$ with weight $\mu_j, \nu_j > 0$, $j \in J$ such that

$$\mu \geq |\nabla u|^2 dx + \sum_{j \in J} \mu_j \, \delta_{\{x = x^{(j)}\}}$$

$$\nu \geq |u|^{2^*} dx + \sum_{j \in J} \nu_j \, \delta_{\{x = x^{(j)}\}}$$

with

$$S \, \nu_j^{2/2^*} \leq \mu_j \qquad \forall j \in J$$

$$\sum_{j \in J} \nu_j^{2/2^*} < \infty, \quad \sum_{j \in J} \mu_j$$

Remark: An analogous result holds for the embedding

$$\dot{W}^{k,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$$

when

$$k \cdot p < n, \quad p > 1$$

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$$

(Here

$$\overset{\circ}{W}^{k,p}(\mathbb{R}^n) = \operatorname{clos}(C_c^\infty(\mathbb{R}^n))$$

with respect to $\|u\|_{\overset{\circ}{W}^{k,p}}$

$$= \sum_{|\alpha| = k} \|\partial^\alpha u\|_{L^q}).$$

Proof of theorem 5.4:

(i) Normalization:

Let $v_k = u_k - u \overset{\omega}{\rightarrow} 0$ in $\dot{H}^1(\mathbb{R}^n)$

Note that

$$|v_k|^{2^*} dx = |u_k|^{2^*} dx + |u|^{2^*} dx + o(1)$$

with $o(1) \overset{\omega^*}{\rightarrow} 0 \quad (k \to \infty)$ by Lemma 5.3

(iii) Let

$$\nu = \nu_0 + \sum_{j \in J} \nu_j \, \delta_{\{x = x^{(j)}\}} \geq 0$$

non negative

where ν_0 is free of concentration, i.e

$$\forall x_0 \in \mathbb{R}^n: \int_{B_r(x_0)} d\nu_0 \to 0 \quad \text{as } r \to 0.$$

and $\nu_j > 0$ for $j \in J$.

Since $\nu(\mathbb{R}^n) < \infty$, J is at most countable. For $j \in J$ and ~~$0 \leq \zeta \leq 1$ with~~ $\zeta(x^{(j)}) = 1$ we get

$$S \nu_j^{2/2^*} \leq S \cdot \left(\int_{\mathbb{R}^n} |\zeta|^{2^*} d\nu \right)^{2/2^*}$$

$$\leq \int_{\mathbb{R}^n} |\zeta|^2 \, d\mu$$

$\rightsquigarrow$

$$\mu \geq \sum_{j \in J} \mu_j \, \delta_{\{x = x^{(j)}\}}$$

with

$$S \nu_j^{2/2^*} \leq \mu_j \quad \forall j \in J.$$

and

$$S \sum_{j \in J} \nu_j^{2/2^*} \leq \mu(\mathbb{R}^n) < \infty.$$

(iv) Claim: $\nu_0 = 0 \quad \nabla_c$

Proof of the claim:

Let $\Omega \subset \mathbb{R}^n$ be open and $(\zeta_k) \subset C_c^\infty(\Omega)$ with $0 \le \zeta_k \le 1$ be such that

$$\zeta_k(x) \to 1 \qquad \forall x \in \Omega.$$

Then

$$(**) \quad \begin{cases} S(\nu(\Omega))^{2/2^*} = S \cdot \lim_{k\to\infty} \left(\int_{\mathbb{R}^n} |\zeta_k|^{2^*} dx \right)^{2/2^*} \\ \le \lim_{k \to \infty} \int |\zeta_k|^2 \, d\mu \le \mu(\Omega). \end{cases}$$

$\leadsto$ ν is absolutely continuous w.r.t. μ

Radon-Nikodym $\leadsto$ $\exists f \in L^1(\mathbb{R}^n, \mu)$ such that

$$d\nu_0 = f \, d\mu$$

and

$$f(x) = \lim_{r \to 0} \frac{\int_{B_r(x)} d\nu_0}{\int_{B_r(x)} d\mu}$$

μ-a.e. $x \in \mathbb{R}^n$

But by (**) we get

$$\frac{\int_{B_r(x)} d\nu_0}{\int_{B_r(x)} d\mu} = \frac{\left(\int_{B_r(x)} d\nu_0\right)^{2/2^*}\left(\int_{B_r(x)} d\nu_0\right)^{(2^*-2)/2^*}}{\int_{B(x)} d\mu}$$

$$\le \frac{1}{S}\left(\int_{B_r(x)} d\nu_0\right)^{\overbrace{(2^*-2)/2^*}^{>0}} \xrightarrow[r\to 0]{} 0 .$$

Hence, $f(x)=0$ μ-a.e. $x \in \mathbb{R}^n$

and thus $\nu_0 = 0$

// Claim

// Thm