Wave equations

Prof. Dr. Tobias Lamm, Dr. Patrick Breuning
tobias.lamm@kit.edu, patrick.breuning@kit.edu

Exercise sheet 1
Due: Tuesday, 22.05.2012, during the lecture

1) a) Let \( u : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^2 \)-function. Show that the general solution of the partial differential equation \( \frac{\partial}{\partial x} \frac{\partial}{\partial y} u = 0 \) is
\[
\begin{align*}
  u(x, y) &= F(x) + G(y)
\end{align*}
\]
for arbitrary \( C^2 \)-functions \( F, G : \mathbb{R} \to \mathbb{R} \).

b) Using the change of variables \( \xi = x + t, \eta = x - t \), show
\[
\frac{\partial^2}{\partial t^2} u - \frac{\partial^2}{\partial x^2} u = 0 \quad \text{if and only if} \quad \frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} u = 0.
\]

c) Use a) and b) to rederive d’Alembert’s formula.

2) Let \( E = (E^1, E^2, E^3), B = (B^1, B^2, B^3) : [t_1, t_2] \times \mathbb{R}^3 \to \mathbb{R}^3 \) solve Maxwell’s equations:
\[
\begin{align*}
  E_t &= \text{curl } B, \\
  B_t &= -\text{curl } E, \\
  \text{div } B &= \text{div } E = 0.
\end{align*}
\]
Show that
\[
\frac{\partial^2}{\partial t^2} u - \Delta u = 0,
\]
where \( u = B^i \) or \( E^i \) for \( i = 1, 2, 3 \).

Remark: The curl of a vector field \( F : \mathbb{R}^3 \to \mathbb{R}^3 \) is defined by
\[
\text{curl } F = (\partial_y F^3 - \partial_z F^2, \partial_z F^1 - \partial_x F^3, \partial_x F^2 - \partial_y F^1).
\]
Here, for \( E \) and \( B \), the curl is calculated with respect to the three space variables.

3) For all \( k \in \mathbb{N} \) let \( \phi \in C^{k+1}(\mathbb{R}) \). Show the following identities:

a) \[
\left( \frac{d^2}{dr^2} \right) \left( \frac{1}{r} \frac{d}{dr} \right)^{k-1} (r^{2k-1} \phi(r)) = \left( \frac{1}{r} \frac{d}{dr} \right)^{k} \left( r^{2k} \frac{d}{dr} \phi(r) \right).
\]

b) \[
\left( \frac{1}{r} \frac{d}{dr} \right)^{k-1} (r^{2k-1} \phi(r)) = \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{d^j \phi}{dr^j}(r),
\]
where the constants \( \beta_j^k \) (\( j = 0, \ldots, k - 1 \)) are independent of \( \phi \).

c) \( \beta_0^k = 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k - 1) \).