\(1) \quad \lim_{n \to \infty} N^{-1} e^{-\frac{n}{2}} = 0 \quad \Rightarrow \exists \ N_0 \forall n > N_0 : N^{-1} e^{-\frac{n}{2}} < 1 \quad \Rightarrow \quad N^{-1} < e^{\frac{1}{2}}

\Gamma(x) = \int_{0}^{\infty} x^{-1} e^{-x} \, dx = \int_{0}^{\infty} x^{-1} e^{-x} \, dx + \int_{\infty}^{0} x^{-1} e^{-x} \, dx \leq \int_{0}^{\infty} x^{-1} \, dx + \int_{0}^{\infty} e^{-\frac{x}{2}} \, dx

\Gamma(x) \leq \left[ \frac{N^x}{x} \right]_0^{\infty} + [-2e^{-\frac{1}{2}}] = \frac{N^x}{x} + 2e^{-\frac{1}{2}} < \infty \quad \forall x > 0

\text{ii) } \quad \text{Induktion:}

\text{I.a: } n = 0 : \Gamma(1) = \int_{0}^{\infty} N^0 e^{-N} \, dN = \left[ -e^{-\lambda} \right]_0^{\infty} = 1 = 0!

\text{I.v: } \exists n_0 : \Gamma(n_0 + 1) = n_0!

\text{I.s: } \Gamma(n_0 + 1) = n_0! \quad \Rightarrow \quad \Gamma(n_0 + 1) + 1 = (n_0 + 1)!

\begin{align*}
\Gamma(n + 2) & = \int_{0}^{\infty} x^{-1} e^{-x} \, dx = \left[ f = x^{n+1} \rightarrow f' = (n+1)x^n \right] = \left[ -x^{n+1} e^{-x} \right]_0^{\infty} + (n+1) \int_{0}^{\infty} x^n e^{-x} \, dx \\
& = (n+1) \Gamma(n+1) = (n+1) n! = (n+1)!
\end{align*}

\text{x = 0: } f'(x) = n x^{m-1} \sin(x-m) + x^{m-1} \cos(x-m) = g(x)

\text{x = 0: } \quad f'(0) = \lim_{x \to 0} \frac{x \sin(x-m)}{x} = \lim_{x \to 0} x^{m-1} \sin(x-m) = 0 \quad \forall m > 0

\text{und}

\text{x = 0: } \quad \lim_{x \to 0} g(x) = \lim_{x \to 0} \left( n x^{m-2} \sin(x-m) + x^{m-1} \cos(x-m) \right) = 0 \quad \forall n > m + 1

\text{oder}

\lim_{x \to 0} g(x) = 0 \quad \text{für } n = m + 1
2) a) \( \lim_{n \to \infty} a_n = 1 \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : |a_n - 1| < \varepsilon \)

b) \( \sum (-1)^n a_n = \infty \)

c) \( a_n = \frac{1}{2} \) für \( n = 2k \)

d) die Folge vor i) ist nicht monoton

e) \( \forall \varepsilon > 0 \exists S > 0 \forall x, y : |x-y| < S \Rightarrow |f(x) - f(y)| < 3 \\
|\frac{1}{1+x^2} - \frac{1}{1+y^2}| = \left| \frac{(x+y)(x-y)}{(1+x^2)(1+y^2)} \right| \leq |x-y| \left( \frac{x}{1+x^2} + \frac{y}{1+y^2} \right) \leq 2|x-y| \Rightarrow \forall \varepsilon > 0 \forall x, y : |x-y| < \frac{3}{2} \Rightarrow |f(x) - f(y)| < 3 \\
0 \leq \left| \frac{x}{1+x^2} \right| = 1 \forall x \in \mathbb{R} \ \\
x \in (0, 1) \quad \frac{x}{1+x^2} \leq 1 - \frac{1}{x} \leq 1 \ \\
x \in (-\infty, 0) \quad \frac{x}{1+x^2} \leq x \leq 1 \\
(\frac{1}{1+x^2})' = \frac{2x(i+x)^2}{} = 0 \quad x = 0 \\
f'(x) > 0 \quad x < 0 \quad \Rightarrow \quad f(0) = 1 \quad \text{Minimum} \\
f'(x) < 0 \quad x > 0 \\
\lim_{x \to -\infty} f(x) = 0 \quad \exists x \in \mathbb{R} : f(x) = 0 \Rightarrow f \text{ hat kein Minimum} \\
\lim_{x \to \infty} f(x) = 0 \\
\sum \frac{1}{n+1} = \lim_{n \to \infty} |\frac{1}{n+1}| = \lim_{n \to \infty} \frac{n+1}{n} = 1 \)
\( a(x) \quad K \quad x \in (-R; R) \quad D \quad x \in R \setminus (-R; R) \)

\[ a(1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \quad K \quad \text{Leibnizkriterium} \quad \frac{1}{n} \to 0 \]

\[ a(-1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n} \quad D \quad \text{harmonische Reihe} \]

\[ a(x) \quad K \quad x \in (-1, 1] \]
3) $\forall x \in A \iff |x+1+3i| = |x-3-3i|
\|(x+1)+i\|^2 = \|(x-3)-i\|^2
(x+1)^2 + (y+1)^2 = (x-3)^2 + (y-3)^2
x^2 + 2x + 1 + y^2 + 2y + 1 = x^2 - 6x + 9 + y^2 - 6y + 9
8x - 16 = -8y
y = x + 2

ii) $\forall x \in C \iff x^2 + (y-1)^2 \leq 1 \land (x-1)^2 + (y-2)^2 < 1$

4) Die Funktion auf beschranktes Intervall $[a, b]$ Maximun und Minimum an.

i) $0 < x < 3$
$f(x) = -6x + (3-x^2)$
$f'(x) = -6 + 2x - 10 = -16 + 2x$
$f'(x) = 0 \Rightarrow x = 8$

$x = 0 \mid x = 3 \mid x = 4 \mid x = 10$

\begin{array}{c|c|c|c}
& f & & \\
\hline
| & 25 | -14 | -15 | 21 | 25 | -15 |
\end{array}

ii) $f(x) = \ln(x^2 - x + e^{-1})$
$f'(x) = \frac{2x - 1}{x^2 - x + e^{-1}}$
$f'(x) = 0 \Rightarrow x = \frac{1}{2}$

\begin{array}{c|c|c|c}
& f & & \\
\hline
| x = 1 | x = 3 | & \\
\hline
-1 & \ln(6 + e^{-1}) & & \\
\hline
\end{array}

Max: \ln(6 + e^{-1})
Min: -1
4) a) \( f(x) = \ln(1+x) \)
\[
\frac{f(x)}{x} = 1 - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^5)
\]
\[
g(x) = \cos x
\]
\[
g(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + o(x^{10}) = 1 + y(x) + o(x^{10})
\]
\[
f(x) = g(x) - \frac{(g(x))^2}{2} + \frac{(g(x))^3}{3} - \frac{(g(x))^4}{4} + o(x^{10})
\]
\[
= \left( -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \right) - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + o(x^{10})
\]
\[
x \cdot \left( -\frac{1}{2} \right) + x^4 \left( \frac{1}{4!} - \frac{1}{2} \left( \frac{1}{2!} \right)^2 \right) + x^6 \left( -\frac{1}{6!} - \frac{1}{2} \left( -\frac{1}{4!} \right) \right) + x^8 \left( \frac{1}{8!} - \frac{1}{2} \left( -\frac{1}{4!} \right) \right)
\]
\[
h) \quad \sin(\sqrt{x}) - \sin(\sqrt{x+1}) = \frac{1 \cos \sqrt{x}}{2 \sqrt{x}} (\sqrt{x} - \sqrt{x+1})
\]
\[
= \frac{1}{2} \cos \sqrt{x} \frac{-1}{\sqrt{x} (\sqrt{x} + \sqrt{x+1})}
\]
\[
|\sin(\sqrt{x}) - \sin(\sqrt{x+1})| = \frac{1}{2} \cos \frac{\sqrt{x}}{\sqrt{x+1}} \frac{-1}{\sqrt{x} (\sqrt{x+1})} \implies \lim_{x \to \infty} (\sin \sqrt{x} - \sin \sqrt{x+1}) = 0
\]
ii) \[
\lim_{x \to 0} \frac{\ln(\cos(3x))}{\ln(\ln(2x))} = \frac{3}{2} \lim_{x \to 0} \frac{\sin 3x}{2 \sin 2x} \lim_{x \to 0} \frac{\cos 3x}{2 \cos 2x} = \frac{3}{2} \lim_{x \to 0} \frac{\cos 3x}{2 \cos 2x} = \frac{9}{4}
\]