Global solutions of the 1D cubic NLS for initial data in $M_{p,q}(\mathbb{R})$

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Statement of the result

Consider the Cauchy-problem for the one-dimensional cubic nonlinear Schrödinger equation

$$\mathrm{i}u_t = u_{xx} \pm |u|^2 u \quad (x,t) \in \mathbb{R}\times\mathbb{R} \qquad \text{and} \qquad u(\cdot, t=0) = u_0. \tag{1}$$

Theorem 1 (to appear in Journal of Differential Equations, [CHKP17]). *Let $p \in \left[2, \frac{35}{17}\right)$ and $u_0 \in M_{p,p'}(\mathbb{R})$. Then there exists an $r \in (3,4]$ such that for sufficiently large $Q < \infty$, the Cauchy problem* (1) *has a unique global solution $u = v + w$, where*

$$v \in L^{\frac{4r}{r-2}}_{\text{loc}}(\mathbb{R}, L^r(\mathbb{R})) \cap L^\infty_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R})) \quad \text{and} \quad w \in L^Q_{\text{loc}}(\mathbb{R}, M_{r,r'}(\mathbb{R})).$$

Sketch of the proof

- $M_{p,p'} = [L^2, M_{r,r'}]_\theta \hookrightarrow (L^2, M_{r,r'})_{\theta,\infty}$ for $r > p$ and appropriate $\theta \in (0,1)$.
- Norm given by the K-functional

$$\|f\|_{(\theta,\infty)} = \sup_{t>0}\left(t^{-\theta}\inf_{f=g+h}\left[\|g\|_2 + t\,\|h\|_{M_{r,r'}}\right]\right) \lesssim \|f\|_{M_{p,p'}} =: c_0.$$

- Put $\alpha = \frac{\theta}{1-\theta}$ and for $N > 0$ set $t = N^{\alpha+1}$. This allows for splitting

$$u_0 = \phi_0^N + \psi_0^N, \quad \left\|\phi_0^N\right\|_2 \leq c_0 N^\alpha, \quad \text{and} \quad \left\|\psi_0^N\right\|_{M_{r,r'}} \leq \frac{c_0}{N}.$$

- Consider the nonlinear Schrödinger evolution v_0 of ϕ_0^N, i.e. the solution of

$$\mathrm{i}v_t = v_{xx} \pm |v|^2 v \quad (x,t)\in\mathbb{R}\times\mathbb{R} \qquad \text{and} \qquad v(\cdot, t=0) = \phi_0^N. \tag{2}$$

  Solution of (2) exists globally and preserves L^2 space norm by [Tsu87].
- *Nonlinear Strichartz estimate* (cf. [HTT11, Lemma 2.4])

$$\|v\|_{L^q([0,\delta], L^r(\mathbb{R}))} \lesssim \left\|\phi_0^N\right\|_2, \qquad \text{where} \qquad \delta \approx \frac{1}{\left\|\phi_0^N\right\|_2^4} \tag{3}$$

  holds for any $2 \leq r \leq 6$, $q > 2$ satisfying *admissibility* $\frac{1}{q} + \frac{1}{2r} = \frac{1}{4}$.
- Consider the corresponding *modified* nonlinear Schrödinger evolution w_0 of ψ_0^N, i.e. solution of

$$\mathrm{i}w_t = w_{xx} \pm G(v_0, w) \quad (x,t)\in\mathbb{R}\times\mathbb{R} \qquad \text{and} \qquad w(\cdot, t=0) = \psi_0^N, \tag{4}$$

  where $G(v,w) = |v+w|^2(v+w) - |v|^2 v$, in $L^Q([0,\delta_N^{(0)}], L^r(\mathbb{R}))$.
- Control on the linear part of (4) by (see [BGOR07] and [CN09])

$$\left\|e^{\mathrm{i}t\partial_x^2}\right\|_{\mathscr{L}(M_{p,q})} \lesssim (1+|t|)^{\left|\frac{1}{2}-\frac{1}{p}\right|} \tag{5}$$

  and $M_{r,r'} \hookrightarrow L^r(\mathbb{R}^d)$.

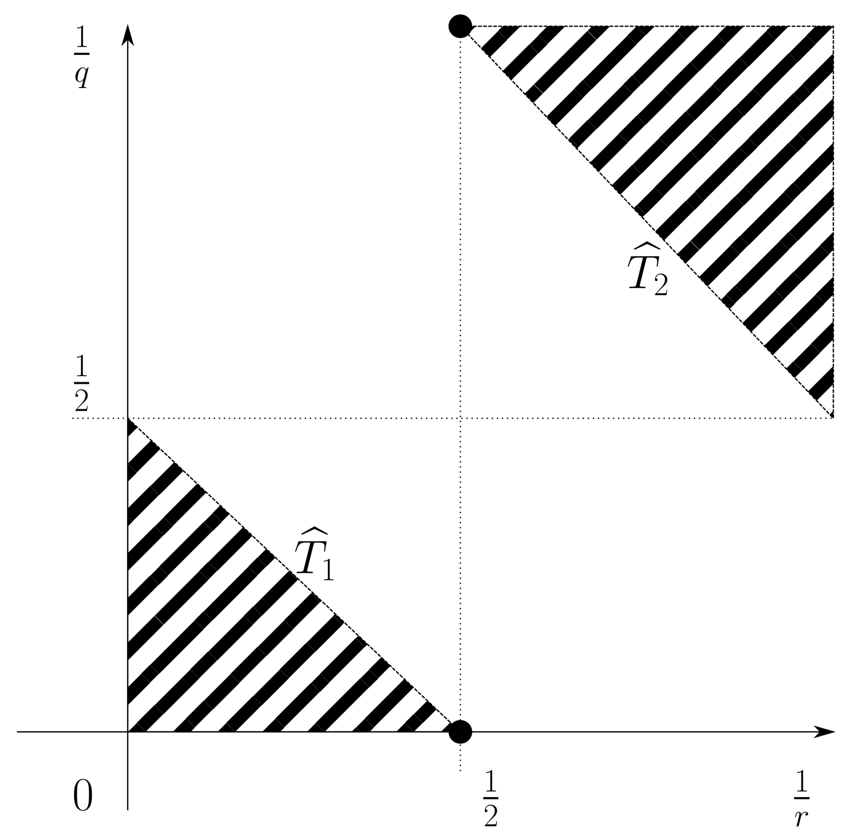
- *Inhomogeneous Strichartz estimate* for *non-admissible pairs* (see [Kat94, Thm. 2.1])

$$\left\|\int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} F(\cdot,\tau)\,\mathrm{d}\tau\right\|_{L^q([0,T],L^r(\mathbb{R}))} \lesssim_{q,r} \|F\|_{L^\gamma([0,T],L^\rho(\mathbb{R}))} \tag{6}$$

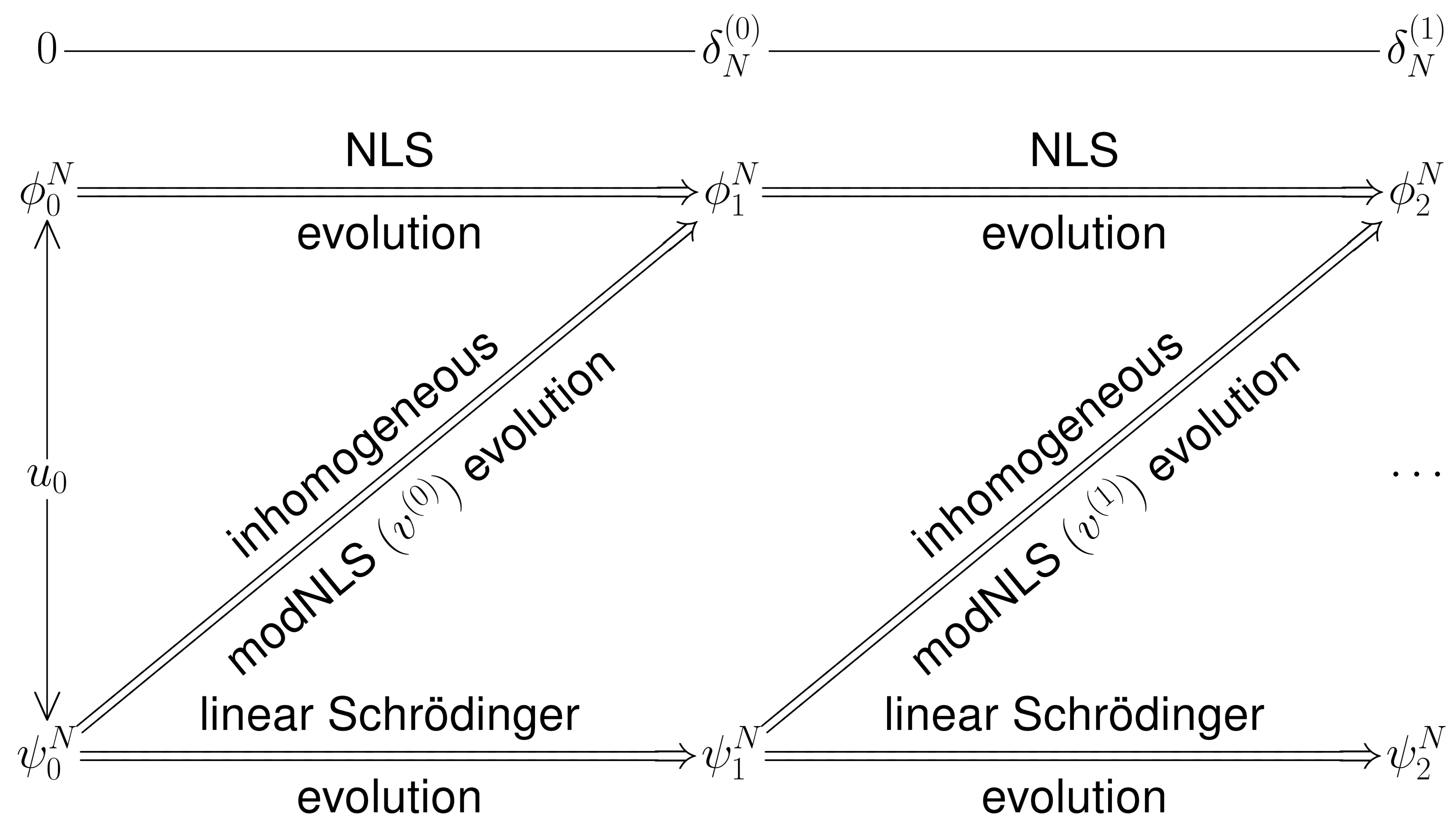
  holds for any $\left(\frac{1}{r},\frac{1}{q}\right) \in \widehat{T}_1$ and $\left(\frac{1}{\rho},\frac{1}{\gamma}\right) \in \widehat{T}_2$, if $\left(\frac{1}{\rho}+\frac{2}{\gamma}\right) - \left(\frac{1}{r}+\frac{2}{q}\right) = 2$, where

$$\widehat{T}_1 = \left\{(x,y) \in \left(0,\frac{1}{2}\right)^2 \,\middle|\, y < \frac{1}{2} - x\right\} \cup \left\{\left(\frac{1}{2}, 0\right)\right\} \quad \text{and}$$

$$\widehat{T}_2 = \left\{(x,y) \in \left(\frac{1}{2},1\right)^2 \,\middle|\, y > \frac{3}{2} - x\right\} \cup \left\{\left(\frac{1}{2}, 1\right)\right\}.$$



- Control on the $L^Q([0,\delta_N^{(0)}], L^r(\mathbb{R}))$-norm of the nonlinear part of (4) by (6) together with (3) applied to $F = \tilde{G}(v_0, w_1, w_2) := G(v_0, w_1) - G(v_0, w_2)$ yields local well-posedness.
- Similarly obtaining control of the $L^\infty([0,\delta_N^{(0)}], L^2(\mathbb{R}))$-norm of the nonlinear part of (4) allows to reiterate



with

$$\phi_1^N = v_0(\cdot,t) + \int_0^t e^{\mathrm{i}(t-\tau)\partial_x^2} G(v_0, w_0)\,\mathrm{d}\tau\Big|_{t=\delta_N^{(0)}}, \qquad \psi_1^N = e^{\mathrm{i}t\partial_x^2}\psi_0^N\Big|_{t=\delta_N^{(0)}}.$$

- Repeat this procedure as long as a certain bound on $\left\|\phi_k^N\right\|_2$ holds, say $K(N)+1$ times. This yields solution of (1) up to time $T(N) = \sum_{k=0}^{K(N)} \delta_N^{(k)}$. Calculations, under the assumption $\alpha \in \left(0, \frac{6r - r^2}{22r^2 - 38r + 12}\right)$ and Q large enough, show $T(N) \to \infty$ as $N \to \infty$, i.e. global solution.

Some remarks

- Proof is inspired by [HT12]. There however, many quantities were conserved from one step to the other.
- Other results involving splitting are [VV01], [Grü05]. Splitting method itself goes back to [Bou99], at least.
- Theorem 1 remains true in higher dimensions and nonlinearities of the form $|u|^{p-1}u$ for $1 < p < 1 + \frac{4}{d}$ under adjustments of the range of r and α.
- All other relevant global existence results require small $M_{p,q}$-norm: See [WH07] and [Kat14] (also [WHHG11] and [RSW12]).

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