Global solutions of the 1D cubic NLS for initial data in $M_{p,q}(\mathbb{R})$

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Statement of the result

Consider the Cauchy-problem for the one-dimensional cubic nonlinear Schrödinger equation

$$u_{n} = u_{xx} + |u|^{2}u \quad (x,t) \in \mathbb{R} \times \mathbb{R} \quad \text{and} \quad u(x,t) = 0, \quad t = 0$$

(1)

**Theorem 1** (to appear in Journal of Differential Equations, [CHRPT17].) Let $p \in [2, \frac{6}{4}]$ and $w_{0} \in M_{p,q}(\mathbb{R})$. Then there exists an $r \in (3, 4)$ such that for sufficiently large $q < \infty$, the Cauchy problem (1) has a unique global solution $u = v + w$, where

$$v \in L^{r}_{loc}(\mathbb{R}, L^{r}((\mathbb{R})) \cap L^{\infty}_{loc}(\mathbb{R}, L^{r}(\mathbb{R})) \quad \text{and} \quad w \in L^{r}_{loc}(\mathbb{R}, M_{r}(\mathbb{R}))$$

Sketch of the proof

- Let $M_{p,q}(\mathbb{R}) = \{L^{p}(\mathbb{R}), M_{q}(\mathbb{R})\}$ for $r > p$ and appropriate $\theta \in (0, 1)$.
- For $r > p$, the solution of (1) exists globally and preserves $L^{r}$ space norm by [WH07].
- Consider the nonlinear Schrödinger evolution $v_{n}$ of $v_{0}$, i.e. the solution of $v_{n} = v_{n+1}$.
- Solution of (1) exists globally and preserves $L^{r}$ space norm by [WH07].
- Similarly obtaining control of the $L^{r}(0, \delta^{p}, L^{r}(\mathbb{R}))$-norm of the nonlinear part of (1) by (5) yields local well-posedness.
- Control of the linear part of (1) by (see [BGG07] and [GN09])

$$\left\|v_{0}\right\|_{L^{r}(\mathbb{R})} \leq \left\|\phi_{n}\right\|_{L^{r}(\mathbb{R})} \leq 1$$

(3)

$$\int_{0}^{t} \int_{\mathbb{R}} e^{i\tau x} F(t, x) dx \leq \int_{0}^{t} \int_{\mathbb{R}} F(t, x) dx$$

(6)

**Inhomogeneous Strichartz estimate for non-admissible pairs** (see [KST06], Thm. 2.1)

$$\int_{0}^{t} \int_{\mathbb{R}} e^{i\tau x} F(t, x) dx \leq C$$

(6)

**Repeat this procedure as long as a certain bound on $\left\|\phi_{n}\right\|_{L^{r}}$ holds, say $N(N') + 1$ times.** This yields solution of (1) up to time $T(N) = \sum_{k=0}^{N'} 2^{k}$.

Calculations, under the assumption $a \in (\frac{1}{2})$ and $Q$ large enough, show $T(N) \to \infty$ as $N \to \infty$, i.e. global solution.

Some remarks

- Proof is inspired by [HT12]. There however, many quantities were conserved from one step to the other.
- Other results involving splitting are [VV01], [GGR05]. Splitting method itself goes back to [GG05], at least.
- Theorem 1 remains true in higher dimensions and nonlinearities of the form $|u|^{q} u$ for $1 < p < 1 + \frac{4}{d}$ under adjustments of the range of $r$ and $\alpha$.
- All other relevant global existence results require small $M_{p, q}$-norm: See [WH07] and [KBT] (also [WHHST] and [RBS17]).

References


