I declare that, to the best of my knowledge, the material contained in this dissertation is original and my own work, except where otherwise indicated, cited or commonly known.

University of Warwick, September 13, 2009
Abstract

A stationary 1-varifold on the plane has a naturally related $\mathbb{R}^{2\times2}$-valued Radon measure. In this dissertation it is shown that this Radon measure is the second derivative of a convex function. Conversely to every convex function $f : \mathbb{R}^2 \to \mathbb{R}$ there is a stationary 1-varifold on the plane, whose related matrix valued Radon measure coincide with the second derivative of the convex function.

Loosely related to this we present some background material. For example we prove the following well-known facts.

Let $\mu$ be a Radon measure on an open subset $U$ or $\mathbb{R}^n$, there is a sequence $\{f_\epsilon\}_{\epsilon>0} \subset C^\infty(U;\mathbb{R})$ such that $f_\epsilon \mathcal{L}^n \rightharpoonup \mu$ in the sense of Radon measures.

Let $\mu$ be a non-negative Radon measure on $\mathbb{R}^n$, $\nu : \mathbb{R}^n \to \mathbb{R}^n \mu$-measurable with $|\nu| = 1$ $\mu$-a.e satisfying $\int g \cdot \sigma \, d\mu = 0$ $\forall g \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ with $\text{div} \, g = 0$. Then there is a BV function $f : \mathbb{R}^n \to \mathbb{R}$ such that $[Df] = \nu \mu$. 
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Acknowledgments

My dissertation could not have been written without the generous help and intellectual support of many people, to whom I extend my gratitude.

First of all I want to thank David Preiss for being an incredibly inspiring supervisor. I benefited greatly from his support and his patience with me, particularly in grasping new and complicated ideas. I owe much of my fascination of geometric measure theory to his invaluable insights.

Furthermore, I just want to thank everyone in the Mathematics Institute for making this year so special to me. Especially to those who helped me when I was lost, gave me advice or happened to be there when I wanted someone to talk through my ideas with.
Introduction

The main result of this dissertation states that stationary 1-varifolds in the plane have a representation given by the second derivative of convex functions. The further presentation deals with background material that is loosely related to the actual proof and helps to clarify it.

An outline of this work is as follows. Chapter 1 quotes the basics of the theory of varifolds needed in the following. They are stated without proofs. For a general treatment of the theory of varifolds the reader is referred to Chapter 4 and 8 of Simon’s book [Sim83] or Allard’s original work [All72]. Other introductions are the notes of Simon [HS86] or Allard [All87]. They are less detailed and focus on the basic and most fundamental theorems, mostly sketching the proofs and main ideas. We further clarify the relation between n-varifolds with co-dimension 1 and vector valued measures. Finally we state our theorem about the representation.

Chapter 2 is just loosely related to the actual proof of the representation theorem. It covers some background material and presents original proofs for well-known statements that are considered as common sense. It is mainly dealing with the question: under which circumstances is a Radon measure on \( \mathbb{R}^n \) the derivative of a BV function. Furthermore we consider the approximation of Radon measures by smooth functions.

Chapter 3 establishes estimates for convex functions, needed for the actual proof of the representation theorem. Again they are known in general but we present original proofs.

Chapter 4 finally presents the owed proof of the main theorem and the corollary stated in the first section. In fact, we give two proofs for a part of the theorem, the first one uses purely calculus methods whilst the second is more geometric. This gives a deeper insight into how the second derivatives of convex functions represent varifolds. The drawback of this is that it only holds when we work in the plane, since the key calculations cannot be directly generalised to higher dimensions.

The notation used is taken from [EG92] and extended in the varifold context by the
notation used by Leon Simon in [Sim83].

Some notation that is not consistently denoted or is less common:

$B(x, r)$ is denoting the open ball centred at $x$ with radius $r$. If it is not otherwise indicated, $U$ is an open subset of $\mathbb{R}^n$ or $\mathbb{R}^{n+1}$ depending on the given context.

$G(n, k)$ denotes the Grassmann manifold or Grassmanian, the space of $k$ dimensional subspaces of $\mathbb{R}^n$. 
Chapter 1

Stationary varifolds of co-dimension 1

The classical definition of a co-dimension 1 varifold is

**Definition 1** (co-dimension 1 varifold). An \( n \)-varifold of co-dimension 1 is a nonnegative Radon measure \( \nu \) on \( U \times G(n+1,n) \), where \( U \) denotes an open subset of \( \mathbb{R}^{n+1} \).

In the case of co-dimension 1, we have \( G(n+1,n) \cong G(n+1,1) \cong S^n/P \) where \( P : S^n \to S^n \) by \( Px = -x \). Thus we can think of measures on \( G(n+1,n) \) as measures on \( S^n/P \). Therefore the co-dimension 1 varifolds are precisely the Radon measures on \( U \times S^n/P \).

The stationary condition is consequently

**Theorem 2** (stationary varifold). An \( n \)-varifold \( \nu \) is stationary if and only if for every \( C^1 \)-vector field \( \psi : U \to \mathbb{R}^{n+1} \) with compact support

\[
\int_{U \times S^n/P} \text{div}_\pi \psi(x) \, d\nu(x,\pi) = 0,
\]

where \( \text{div}_\pi \psi(x) := \sum_{i=1}^{n} \tau_i \cdot \nabla_{\tau_i} \psi(x) \) for an orthonormal basis \( \{\tau_i\}_{i=1}^{n} \) of \( \text{span}(\pi) \perp \).

\( \nabla_v \psi(x) := \lim_{t \to 0} \frac{1}{t}(\psi(x+tv) - \psi(x)) \) for any \( v \in S^n \).

It turns out that the stationary condition is just a condition at the ”second moment” of the Radon measure \( \nu \).

**Definition 3** (second moment of a varifold). The second moment of an \( n \)-varifold \( \nu \) is a \( \mathbb{R}^{n+1 \times n+1} \)-valued Radon measure on \( U \) \( (\mu_{i,j}) \) defined as follows

\[
\int_U \varphi(x) \, d\mu_{i,j} := \int_{U \times S^n/P} \varphi(x) \, \pi_i \pi_j \, d\nu(x,\pi)
\]

for all \( \varphi \in C_c(U;\mathbb{R}) \) and \( \pi = (\pi_1,\ldots,\pi_{n+1}) \in S^n/P \).
Using this definition the stationary condition translates to

**Lemma 4** (stationary condition for varifolds of co-dimension 1). A varifold $\nu$ with second moment $(\mu_{i,j})$ is stationary if and only if for every compact supported $C^1$-vector field $\psi = (\psi_1, \ldots, \psi_{n+1}) : U \to \mathbb{R}^{n+1}$

$$\sum_{i,j=1}^{n+1} \left( \int_U \frac{\partial \psi_i}{\partial x_i} \, d\mu_{j,j} - \int_U \frac{\partial \psi_i}{\partial x_j} \, d\mu_{i,j} \right) = 0. \quad (1.3)$$

**Proof.** For a $C^1$-vector field $\psi : U \to \mathbb{R}^{n+1}$ we have by definition $\nabla_v \psi(x) = D\psi(x)v$ for all $v \in S^n$. For any $\pi \in S^n \pi \otimes \pi$ is the orthogonal projection onto span($\pi$) and $1 - \pi \otimes \pi$ is therefore the orthogonal projection onto span($\pi$)$^\perp$. Let $\{\tau_1, \ldots, \tau_n, \tau_{n+1}\}$ be an orthonormal basis for span($\pi$)$^\perp$ then $\{\tau_1, \ldots, \tau_n, \pi\}$ is an orthonormal basis for $\mathbb{R}^{n+1}$ with $(1 - \pi \otimes \pi)\tau_i = \tau_i$, $(1 - \pi \otimes \pi)\pi = 0$. As result we can write

$$\text{trace}((1 - \pi \otimes \pi)D\psi(x)) = \sum_{i=1}^n \tau_i \cdot (1 - \pi \otimes \pi)D\psi(x)\tau_i + \pi \cdot (1 - \pi \otimes \pi)D\psi(x)\pi$$

$$= \sum_{i=1}^n \tau_i \cdot D\psi(x)\tau_i = \sum_{i=1}^n \tau_i \cdot \nabla_{\tau_i} \psi(x)$$

$$= \text{div}_\pi \psi(x).$$

We can evaluate the trace as well using the canonical basis of $\mathbb{R}^{n+1}$, hence $\text{trace}((1 - \pi \otimes \pi)D\psi(x)) = \sum_{i=1}^{n+1} \frac{\partial \psi_i}{\partial x_i} - \sum_{i,j=1}^n \frac{\partial \psi_i}{\partial x_j}\pi_j\pi_i$. Together with $1 = \sum_{j=1}^{n+1} (\pi_j)^2$ we can rewrite the stationary condition (1.1) to

$$\int_{U \times S^n / P} \text{div}_\pi \psi(x) \, d\nu(x, \pi) = \int_{U \times S^n / P} \sum_{i,j=1}^{n+1} \frac{\partial \psi_i}{\partial x_i}(\pi_j)^2 \, d\nu(x, \pi) - \sum_{i,j=1}^{n+1} \int_{U \times S^n / P} \frac{\partial \psi_i}{\partial x_j}\pi_j\pi_i \, d\nu(x, \pi)$$

$$= \sum_{i,j=1}^{n+1} \int_{U \times S^n / P} \frac{\partial \psi_i}{\partial x_i}(\pi_j)^2 \, d\nu(x, \pi) - \sum_{i,j=1}^{n+1} \int_{U \times S^n / P} \frac{\partial \psi_i}{\partial x_j}\pi_j\pi_i \, d\nu(x, \pi)$$

$$= \sum_{i,j=1}^{n+1} \int_{U} \frac{\partial \psi_i}{\partial x_i} \, d\mu_{j,j}(x) - \sum_{i,j=1}^{n+1} \int_{U} \frac{\partial \psi_i}{\partial x_j} \, d\mu_{i,j}(x) = 0.$$ 

Based on the following lemma we can think of $n$-varifolds with co-dimension 1 as $\mathbb{R}^{n+1 \times n+1}$ valued Radon measure.

**Lemma 5** (second moment of a varifold). Let $\nu$ be an $n$-varifold with co-dimension 1 i.e. a Radon measure on $U \times S^n / P$. Then its second moment $(\mu_{i,j})$ is a positive
semidefinite symmetric matrix valued Radon measure i.e.

\[ 0 \leq \sum_{i,j=1}^{n+1} \int_U \psi_i \psi_j \, d\mu_{i,j} \quad \text{for every vector field } \psi = (\psi_1, \ldots, \psi_{n+1}) \in C_c(U; \mathbb{R}^{n+1}) \]

(1.4)

\[ \int_U \varphi \, d\mu_{i,j} = \int_U \varphi \, d\mu_{j,i} \quad \text{for all } i, j = 1, \ldots, n + 1 \text{ and } \varphi \in C_c(U; \mathbb{R}) \]

Conversely if \((\mu_{i,j})\) is a \(\mathbb{R}^{n+1 \times n+1}\)-valued Radon measure satisfying (1.4), then there is at least one \(n\)-varifold \(\nu\) with second moment \((\mu_{i,j})\).

**Proof.** Let \(\nu\) be an \(n\)-varifold then (1.4) holds, since

\[ \sum_{i,j=1}^{n+1} \int_U \psi_i \psi_j \, d\mu_{i,j} = \sum_{i,j=1}^{n+1} \int_{U \times S^n/P} \psi_i \psi_j \pi_i \pi_j \, d\nu(x, \pi) = \int_{U \times S^n/P} (\psi \cdot \pi)^2 \, d\nu(x, \pi) \geq 0. \]

The symmetry condition follows directly from the definition.

The converse is mainly an extension problem of positive linear functional as considered for example in [MD00]. Although we could rely on such general theorems, for the sake of completeness we will present a simpler proof that is sufficient in our context. Note, however, that this simpler proof is inspired by many of the proofs of these general results.

We will show that in our situation there exists a positive extension whenever a certain inequality is satisfied. Subsequently we will check that (1.4) implies this inequality.

The set of continuous functions with compact support \(C_c(U \times S^n/P; \mathbb{R})\) is a real valued vector space. Endowed with the supremum norm, \(\|\varphi\| = \sup |\varphi|\), it becomes a normed vector space. Furthermore for any compact subset \(K\) of \(U\) the subset \(\{\varphi \in C_c(U \times S^n/P; \mathbb{R}) : \text{supp}(\varphi) \subset K \times S^n/P\}\) is a Banach space.

We observe that the following map on \(C_c(U \times S^n/P; \mathbb{R})\)

\[ \varphi \mapsto \sup(-\varphi) \]

is a non-negative sublinear functional with \(\sup(-\varphi) = 0\) if and only if \(\varphi\) is non-negative. Actually \(\sup(-\varphi)\) is the Minkowski functional of the convex set \(\mathcal{P} - 1\), where \(\mathcal{P}\) is the set of all non-negative functions in \(C_c(U \times S^n/P; \mathbb{R})\).

**Claim #1.** Let \(\mathcal{C}\) be a subspace of \(C_c(U \times S^n/P; \mathbb{R})\). A continuous linear functional \(l\) on \(\mathcal{C}\) has a positive extension \(L\) if for any compact subset \(K \subset U\) there is a constant \(C(K) > 0\) s.t

\[ -l(\varphi) \leq C(K) \sup(-\varphi) \quad \text{for all } \varphi \in \mathcal{C} \text{ with } \text{supp}(\varphi) \subset K \times S^n/P. \]

(1.5)

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In fact this condition is not only sufficient but necessary.

Proof of Claim #1. Firstly we observe that the inequality implies continuity. For any non-negative \( \phi \) with \( \text{supp}(\phi) \in K \times S^n/P \) \([1.5]\) implies \( 0 \leq l(\phi) = -l(-\phi) \leq C(K)\|\phi\| \). For any \( \phi \in C_c(U \times S^n/P; \mathbb{R}) \) with \( \text{supp}(\phi) \subset K \times S^n/P \) we have \( \phi = \phi^+ - \phi^- \) with \( \phi^+ = \max\{\phi, 0\}, \phi^- = \max\{-\phi, 0\} \) non-negative and therefore

\[
|l(\phi)| \leq |l(\phi^+)| + |l(\phi^-)| \leq C(K)(\|\phi^+\| + \|\phi^-\|) \leq 2C(K)\|\phi\|.
\]

Take a sequence \( \{K_m\} \) of compact subsets of \( U \) s.t. \( K_m \subset K_{m+1} \) and \( K_m \not\subset U \) as \( m \to \infty \). Set \( C_0 := C, C_m := C \cup \{\phi \in C_c(U \times S^n/P; \mathbb{R}) : \text{supp}(\phi) \subset K_m\} \) and \( l_0 := l \).

We will prove now that if \( l_m \) is a continuous linear functional on \( C_m \) satisfying \([1.5]\) then there is an extension \( l_{m+1} \) on \( C_{m+1} \) preserving this inequality. By induction the claim will be proven.

Let \( l_m \) satisfy \([1.5]\) then \( \phi \mapsto C(K_{m+1}) \sup(-\phi) \) is a sublinear functional on \( C_c(K_{m+1} \times S^n/P; \mathbb{R}) \) and by assumption \( -l(\phi) \leq C(K_{m+1})\rho(\phi)\forall \phi \in C_m \) with \( \text{supp}(\phi) \subset K_m+1 \).

Applying the Hahn-Banach Theorem there is a linear functional \( \tilde{l}_{m+1} \) on \( C_c(K_{m+1} \times S^n/P; \mathbb{R}) \) satisfying \( -\tilde{l}_{m+1}(\cdot) \leq C(K_{m+1}) \sup(-\cdot) \) and \( \tilde{l}_{m+1}(\phi) = l_m(\phi) \) for all \( \phi \in C_m \cap \{\phi : \text{supp}(\phi) \subset K_{m+1}\} \). Therefore

\[
l_{m+1}(\phi) := \begin{cases} l_m(\phi), & \text{if } \phi \in C \\ \tilde{l}_{m+1}(\phi), & \text{if } \text{supp}(\phi) \subset K_{m+1} \end{cases}
\]

is a well-defined continuous linear functional on \( C_{m+1} \) satisfying \([1.5]\) and extending \( l_m \).

By induction and the fact \( C_c(U \times S^n/P; \mathbb{R}) = \bigcup_n C_m \) there is a continuous linear functional \( L \) on \( C_c(U \times S^n/P; \mathbb{R}) \) extending \( l \) i.e. \( L \upharpoonright C = l \). Let \( \phi \in \mathcal{P} \) with \( \text{supp}(\phi) \subset K \) for some compact set \( K \subset U \) then

\[
-L(\phi) \leq C(K) \sup(-\phi) = 0.
\]

Thus \( L \) is a positive linear functional and by Riesz Representation Theorem there is a Radon measure \( \nu \) on \( U \subset S^n/P \) with

\[
L(\phi) = \int_{U \times S^n/P} \phi \, d\nu \text{ for all } \phi \in C_c(U \times S^n/P; \mathbb{R}).
\]

It remains to check that \([1.4]\) implies the sufficient condition.

Claim #2. A \( \mathbb{R}^{n+1 \times n+1} \) - valued Radon measure \( (\mu_{i,j}) \) satisfying \([1.4]\) fulfils \([1.5]\)
Furthermore \( \mu = \sum_{i,j=1}^{n+1} a_{i,j} \pi_i \pi_j \) is a symmetric positive semidefinite matrix with trace \( \sigma = \sum_{i,j=1}^{n+1} \sigma_{i,j} \). Based on the fact that for every symmetric positive semidefinite matrix \( \phi \) with 
\[
\sum_{i,j=1}^{n+1} \phi_{i,j} \]
that 
\[
\text{Fix } \pi \text{ such that } O \text{ is positive semidefinite for every } x \text{ because } 0 \leq \pi \cdot A(x) \pi \text{ for every } \pi \in S^n.
\]
Thus by classical linear algebra there are continuous maps \( x \mapsto \lambda_i(x) \geq 0 \) for \( i = 1, \ldots, n + 1 \) and \( x \mapsto O(x) \) such that \( O(x) \) is an orthogonal matrix and 
\[
A(x) = O(x) \text{ diag}(\lambda_1(x), \ldots, \lambda_{n+1}(x)) O(x)^t.
\]
This is equivalent to 
\[
a_{i,j}(x) = \sum_{k=1}^{n+1} \lambda_k(x) O_{i,k}(x) O_{j,k}(x).
\]
Set \( \phi^k := \sqrt{\lambda_k}(O_{1,k}, \ldots, O_{n+1,k}) \in C_c(U; \mathbb{R}^{n+1}) \). Thus we obtain 
\[
l(\psi) = \sum_{i,j=1}^{n+1} \int_U a_{i,j} \, d\mu_{i,j} = \sum_{k=1}^{n+1} \sum_{i,j=1}^{n+1} \int_U \phi^k_{i,j} \phi^k_{j,i} \, d\mu_{i,j} \geq 0.
\]
Fix \( K \subset U \) compact and \( \eta_K \in C_c(U; \mathbb{R}) \) with \( 0 \leq \eta_K \leq 1 \) and \( \eta_K \equiv 1 \) on \( K \). Then 
\[
\eta_K = \sum_{i=1}^{n+1} 1_{\eta_i K} \pi_i \pi_i \in C.
\]
It is straightforward to see that for any \( \varphi \in C - P \) with \( \text{supp}(\varphi) \subset K \) that 
\[
\eta_K + \frac{1}{\text{sup}(-\varphi)} \varphi \in C \cap P.
\]
Therefore \( 0 \leq l(\eta_K + \frac{1}{\text{sup}(-\varphi)} \varphi) = l(\eta_K) + \frac{1}{\text{sup}(-\varphi)} l(\varphi) \) equivalent to 
\[
-l(\varphi) \leq C(K) \text{ sup}(-\varphi)
\]
with \( C(K) = l(\eta_K) \). This proves that (1.5) holds since \( -l(\varphi) \leq 0 = C(K) \text{ sup}(-\varphi) \) for any \( \varphi \in C \cap P \).

One can avoid the recursion if one uses instead the differentiation theorem for Radon measures. Based on the fact that for every symmetric positive semidefinite matrix \( S = (s_{i,j}) \in \mathbb{R}^{n+1 \times n+1} \) one has \( s_{i,j} \leq \text{trace} \, S \) one shows that based on (1.4) one has \( \mu_{i,j} \ll \mu := \mu_{1,1} + \mu_{2,2} + \ldots + \mu_{n+1,n+1} \) for all \( i, j = 1, \ldots, n + 1 \). Thus there is a \( \mu \)-measurable matrix \( (\sigma_{i,j}) \) s.t \( \mu_{i,j}(A) = \int_A \sigma_{i,j} \mu \). The construction implies that \( (\sigma_{i,j}) \) is a symmetric positive semidefinite matrix with trace \( \sigma = \sigma_{1,1} + \ldots + \sigma_{n+1,n+1} \) for \( \mu \)-a.e. \( x \). Using the same arguments as above for \( \mu \)-a.e. \( x \) there is a positive Radon measure \( \mu_x \) on \( S^n / P \) extending the linear functional 
\[
l_x(\varphi) := \sum_{i,j=1}^{n+1} a_{i,j} \sigma_{i,j} \text{ for } \varphi(\pi) = \sum_{i,j=1}^{n+1} a_{i,j} \pi_i \pi_j.
\]
Furthermore \( \mu_x(S^n / P) = \sum_{i=1}^{n+1} \int_{S^n / P} (\pi_i)^2 \, d\mu_x = \text{trace}(\sigma) = 1 \) for \( \mu \)-a.e. \( x \). The
The just obtained representation of the Radon measure is in fact the general case. The disintegration theorem shows that every Radon measure \( \nu \) on \( U \times S^n/P \) is equivalent to a Radon measure \( \mu \) on \( U \), probability measure \( \mu_x \) on \( S^n/P \) for \( \mu \)-a.e. \( x \) in \( U \) such that

\[
\int_{U \times S^n/P} \varphi(x, \pi) \, d\nu(x, \pi) = \int_U \int_{S^n/P} \varphi(x, \pi) \, d\mu_x(\pi) \, d\mu(x). \quad \forall \varphi \in C_c(U \times S^n/P; \mathbb{R}).
\]

A proof for this statement can be found in \cite{Sim83} Lemma 38.4.

Therefore the stationary condition (1.3) is equivalent to

\[
\int_U \text{div} \psi(x) \, d\mu(x) - \int_U \text{trace}(D\psi(x)\sigma(x)) \, d\mu(x) = 0, \quad (1.7)
\]

whereas \( \sigma : U \to \mathbb{R}^{n \times n} \) is the positive semidefinite matrix defined as

\[
\sigma_{ij}(x) := \int_{S^{n-1}/P} \pi_i \pi_j \, d\mu_x(\pi) \quad \text{with} \quad \pi = (\pi_1, \ldots, \pi_n) \in S^{n-1}/P.
\]

After we have introduced the necessary notation and explained some relations we can state the main theorem.

**Theorem 6** (Representation of varifolds by second derivatives of convex functions). Let \( U \) be an open convex subset of \( \mathbb{R}^{n+1} \) and \( f : U \to \mathbb{R} \) be any convex function. Its second derivative \( D^2f \) considered as a \( \mathbb{R}^{n+1 \times n+1} \)-valued Radon measure satisfies the stationary condition (1.3) i.e. it is the second moment of a stationary co-dimension 1 varifold.

In case of \( n = 1 \) the converse holds as well. That means that for any stationary 1-varifold \( \nu \) with second moment \( (\mu_{i,j}) \) there is a convex function \( f : U \subset \mathbb{R}^2 \to \mathbb{R} \) with \( D^2f = (\mu_{ij}) \).

The second derivative of a convex function in terms of Radon measure has to be understood in the sense of the following classical theorem.

**Theorem 7** (\cite{EG92} Chapter 6.3, Theorem 2). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be convex. Then there exist signed Radon measures \( \mu_{i,j} = \mu_{j,i} \) such that

\[
\int_{\mathbb{R}^n} f \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \, dx = \int_{\mathbb{R}^n} \varphi \, d\mu_{i,j} \quad (i, j = 1, \ldots, n) \quad (1.8)
\]
for all \( \varphi \in C^2_c(\mathbb{R}^n; \mathbb{R}) \). Furthermore, the measures \( \mu_{i,i} \) are nonnegative for all \( i = 1, \ldots, n \).

For these special class of n-varifolds one can easily establish ”uniqueness and existence of tangent cones” summarised in this corollary.

**Corollary 8** (Unique tangent cones for n-varifolds given by second derivatives of convex functions). Let \( U \) be an open convex subset of \( \mathbb{R}^{n+1} \). Let \( (\mu_{i,j}) \) be the second moment of a stationary co-dimension 1 varifold that agrees with the second derivative of a convex function \( f : U \rightarrow \mathbb{R} \). Then \( (\mu_{i,j}) \) has at every point \( x \in U \) a unique tangent cone. That means for every \( x \in U \) the measures \( \mu_{i,j,x,\lambda}(A) = \lambda^{-n} \mu_{i,j}(x + \lambda A) \) converge to a second moment \( (\nu_{i,j}) \) of a stationary n-varifold that is a cone. In fact \( (\nu_{i,j}) \) is the second derivative of the convex function \( g_x : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) given by

\[
g_x(z) = \lim_{s \searrow 0} \frac{f(x + sz) - f(x)}{s}.
\]

Theorem 6 is sharp in terms of the dimension, meaning that for higher dimensions \( (n > 1) \), the second moment of a co-dimension 1 varifold doesn’t necessarily agree with the second derivative of a convex function. This is shown by the subsequent example.

**Example 1** (Counterexample for higher dimensions - Catenoid). A catenoid, a classical minimal surface, is an example for the \( n = 2 \) case that cannot even locally be represented by the second derivative of a convex function. A catenoid is given in parametric form as the following set in \( \mathbb{R}^3 \)

\[
S = \{ \Phi(\theta, z) := (\cosh z \cos \theta, \cosh z \sin \theta, z) : z \in \mathbb{R}, \theta \in [0, 2\pi) \}.
\]

In respect to the parametric representation the normal is easily calculated to be

\[
n(\theta, z) = \frac{1}{\cosh z} (\cos \theta, \sin \theta, -\sinh z);
\]

hence the catenoid corresponds to the following Radon measure on \( \mathbb{R}^3 \times S^2/P \)

\[
\int_{\mathbb{R}^3 \times S^2/P} \varphi(x, \pi) \, d\nu(x, \pi) := \int_{\mathbb{R}} \int_0^{2\pi} \varphi(\Phi(\theta, z), n(\theta, z)) \cosh^2 z \, dzd\theta.
\]

The second moment of this 2-varifold is

\[
(\mu_{i,j}) = \begin{pmatrix}
\cos^2 \theta & \cos \theta \sin \theta & -\cos \theta \sinh z \\
\cos \theta \sin \theta & \sin^2 \theta & -\sin \theta \sinh z \\
-\cos \theta \sinh z & -\sin \theta \sinh z & \sinh^2 z
\end{pmatrix} \, dzd\theta.
\]
More precisely for \( \varphi \in C_c(\mathbb{R}^3; \mathbb{R}) \) we have
\[
\int_{\mathbb{R}^3} \varphi \, d(\mu_{i,j}) = \int_{\mathbb{R}^3 \times [0,2\pi]} \varphi(\Phi(\theta,z)) \begin{pmatrix}
\cos^2 \theta & \cos \theta \sin \theta & -\cos \theta \sinh z \\
\cos \theta \sin \theta & \sin^2 \theta & -\sin \theta \sinh z \\
-\cos \theta \sinh z & -\sin \theta \sinh z & \sinh^2 z
\end{pmatrix} \, dzd\theta.
\]
(1.12)

The catenoid is a classical minimal surface. The theory of n-varifolds of codimension 1 extends the idea of hypersurfaces in \( \mathbb{R}^{n+1} \) i.e. embedded n-dimensional \( C^k \)-submanifolds of \( \mathbb{R}^{n+1} \). Thus the stationary condition (1.3) has to be satisfied by the catenoid.

Although the above argument is sufficient we are going to present exemplarily the calculations that (1.11) satisfies the stationary condition for vector fields \( \psi = (\varphi,0,0) \), \( \varphi \in C^2_c(\mathbb{R}^3; \mathbb{R}) \). This is in some sense sufficient, since the stationary condition is linear in the components of the test vector fields. Therefore it is sufficient to check the single components separately. In this case we have
\[
\int_{\mathbb{R}^3} \frac{\partial \varphi}{\partial x} \, d\mu_{2,2} + \int_{\mathbb{R}^3} \frac{\partial \varphi}{\partial y} \, d\mu_{3,3} - \int_{\mathbb{R}^3} \frac{\partial \varphi}{\partial y} \, d\mu_{1,2} - \int_{\mathbb{R}^3} \frac{\partial \varphi}{\partial z} \, d\mu_{1,3} = \int_{\mathbb{R}^3 \times [0,2\pi]} \left( \sin^2 \theta + \sinh^2 z \right) \frac{\partial \varphi}{\partial x} \circ \Phi - (\sin \theta \cos \theta) \frac{\partial \varphi}{\partial y} \circ \Phi - (-\cos \theta \sinh z) \frac{\partial \varphi}{\partial z} \circ \Phi \, dzd\theta.
\]

Next we observe that
\[
\frac{d}{dz} (\varphi \circ \Phi) = \cos \theta \sinh z \frac{\partial \varphi}{\partial x} + \sin \theta \sinh z \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial z},
\]
\[
\frac{d}{d\theta} (\varphi \circ \Phi) = -\sin \theta \cosh z \frac{\partial \varphi}{\partial x} + \cos \theta \cosh z \frac{\partial \varphi}{\partial y}.
\]
(1.13)

Use of trigonometric identities \( \cos^2 \theta + \sin^2 \theta = 1 \), \( \cosh^2 z - \sinh^2 z = 1 \) in the first two terms and rearranging appropriate the above is equivalent to
\[
\int_{\mathbb{R}^3 \times [0,2\pi]} \cos \theta \sinh \theta \left( \cos \theta \sinh z \frac{\partial \varphi}{\partial x} + \sin \theta \sinh z \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi}{\partial z} \right) \, dzd\theta
\]
\[
+ \sin \theta \cosh z \left( \sin \theta \cosh z \frac{\partial \varphi}{\partial x} - \cos \theta \cosh z \frac{\partial \varphi}{\partial y} \right) \, dzd\theta
\]
\[
= \int_{\mathbb{R}^3 \times [0,2\pi]} \cos \theta \sinh \theta \frac{d}{dz} (\varphi \circ \Phi) - \sin \theta \cosh z \frac{d}{d\theta} (\varphi \circ \Phi) \, dzd\theta.
\]

Hence by integration by parts
\[
= -\int_{\mathbb{R}^3 \times [0,2\pi]} \cos \theta \cosh z (\varphi \circ \Phi) \, dzd\theta + \int_{\mathbb{R}^3 \times [0,2\pi]} \cos \theta \cosh z (\varphi \circ \Phi) \, dzd\theta = 0.
\]

Similar one can check that the other two components vanish.
Already the figure of a catenoid (see 1.1) is a strong indication that its second moment cannot come from a second derivative of a convex function. If there would be such a function it would have to be affine in the white regions but the intersection between two affine functions is a hyperplane and not something curved. To make this argument rigorous we observe that if a Radon measure \((\mu_{i,j})\) on an open convex subset \(U\) of \(\mathbb{R}^n\) is coming from a second moment of a convex function \(f\) on \(U\) then for any \(\phi \in C^3_c(U; \mathbb{R})\)

\[
\int_U \frac{\partial^2 \phi}{\partial x_i \partial x_j} d\mu_{j,k} - \int_U \frac{\partial^2 \phi}{\partial x_j \partial x_i} d\mu_{i,k} = \int_U \frac{\partial^3 \phi}{\partial x_k \partial x_j \partial x_i} f \, dx - \int_U \frac{\partial^3 \phi}{\partial x_i \partial x_j \partial x_k} f \, dx = 0.
\]

This observation is in the spirit of the fact that every exact covector field is closed. But in case of a catenoid we encounter that for a \(\phi \in C^3_c(\mathbb{R}^3; \mathbb{R})\)

\[
\int_{\mathbb{R} \times [0,2\pi]} \left( \frac{\partial \phi}{\partial x} \circ \Phi \right) (\cos \theta \sin \theta) - \left( \frac{\partial \phi}{\partial y} \circ \Phi \right) (\cos^2 \theta) \, dz d\theta
\]

\[
= \int_{\mathbb{R} \times [0,2\pi]} \frac{\cos \theta}{\cosh z} \frac{d}{d\theta} (\phi \circ \Phi) \, dz d\theta
\]

\[
= - \int_{\mathbb{R} \times [0,2\pi]} \frac{\sin \theta}{\cosh z} (\phi \circ \Phi) \, dz d\theta.
\]

This is in general unequal 0 so that the second moment of a catenoid cannot be the second derivative of a convex function. For example one could choose \(\phi = y \eta(r)\) with \(\eta \in C^\infty_c(\mathbb{R}; \mathbb{R})\) nonnegative, then \(\frac{\sin \theta}{\cosh z} (\phi \circ \Phi) = \sin^2 \theta \eta(\sqrt{\cosh^2 z + z^2})\). Hence the integral does not vanish.
Chapter 2

Radon measures and relations to BV functions

The main theorem that we wish to prove states a relation between functions and Radon measures. A situation where such a relation is well known is the space of functions on $\mathbb{R}^n$ with bounded variations, for short BV functions. It is not our intention to give a survey on this subject. A general account to this can be found in [EG92] or [AFP00].

This subject is interesting in the sense that it provides a feeling for the possible techniques that can be applied in a proof of theorem 6. More precisely, we are interested in finding the general conditions under which a Radon measure on $\mathbb{R}^n$ is the distributional derivative of a BV function. It turns out that the natural ones are already sufficient. In this chapter we present these well known results, together with original proofs.

The approximation of Radon measures by smooth functions, as introduced in lemma 15 and 16 will be of great use. Based on this approximation we will give a second proof of lemma 12 which will contain the main idea of the actual proof of theorem 6. Results will be stated first and then proofs will be given in the subsequent subsection. As already mentioned these results are commonly known.

Throughout this chapter $U$ denotes, as usual, an open subset of $\mathbb{R}^n$.

**Definition 9.** A function $f \in L^1_{\text{loc}}(U; \mathbb{R})$ has locally bounded variation in $U$ if for each open set $V \subset U$,

$$
\sup \left\{ \int_U f \, \text{div} \, \varphi \, dx : \varphi \in C^1_c(V; \mathbb{R}), |\varphi| \leq 1 \right\} < \infty.
$$

We write $\text{BV}_{\text{loc}}(U; \mathbb{R})$ to denote the space of these functions.

The most important implication of this definition is the general structure assertion.
Theorem 10 (Structure Theorem for $BV_{loc}(U; \mathbb{R})$). Let $f \in BV_{loc}(U; \mathbb{R})$. Then there exists a Radon measure $\mu$ on $U$ and $\mu$-measurable function $\sigma : U \to \mathbb{R}^n$ such that

1. $|\sigma(x)| = 1$ for $\mu$-a.e. $x \in U$,
2. $\int_U f \, \text{div} \varphi \, dx = \int_U \varphi \cdot \sigma \, d\mu$ for all $\varphi \in C^1_c(U; \mathbb{R}^n)$.

A proof of this theorem can be found in [EG92] section 5.1. This theorem and the basic definition is the only thing we will need from the theory of BV functions.

Lemma 11. Suppose $\mu$ is a Radon measure on $\mathbb{R}$, then there is a function $f \in L^\infty_{loc}(\mathbb{R})$ such that $Df = \mu$, i.e.

$$
\int_{\mathbb{R}} g' f \, dx = \int_{\mathbb{R}} g \, d\mu \quad \forall g \in C^1_c(\mathbb{R}).
$$

$f$ is determined uniquely up to additive constants.

This lemma gives the answers in the case of $\mathbb{R}$: Every Radon measure on $\mathbb{R}$ is the weak derivative of a $BV_{loc}$ function. In fact one could just define $f : \mathbb{R} \to \mathbb{R}$ as

$$
f(x) := \begin{cases}
\mu([x,0)) & \text{if } x < 0 \\
0 & \text{if } x = 0 \\
-\mu([0,x)) & \text{if } x > 0.
\end{cases}
$$

Fubini’s theorem applied to the function $-g'(x)\chi_{[-R,R]}(t)$, where $R > 0$ large enough such that $\text{supp}(g) \subset (-R,R)$, on the product space $\mathbb{R} \times \mathbb{R}$ with measure $\mathcal{L}^1 \times \mu$ shows that $f$ has the desired properties.

We will now present another proof that already outlines an idea useful for higher dimensions, thus motivating the coming investigation.

Proof. Fix $\eta \in C^\infty(\mathbb{R}; \mathbb{R})$ with $\text{supp}(\eta) \subset (-1,1)$, $0 \leq \eta \leq 1$ and $\int \eta \, dx = 1$. Extending each function $u \in L^1([-m,m]; \mathbb{R})$ by 0 outside of $[-m,m]$ for every $m \in \mathbb{N}$, we can consider $L^1([-m,m]; \mathbb{R})$ as subsets of $L^1(\mathbb{R}; \mathbb{R})$.

It is now straight forward that the map $l : L^1(\mathbb{R}; \mathbb{R}) \to C(\mathbb{R}; \mathbb{R})$

$$
l(u)(x) := \int_{-\infty}^x u(s) \, ds - \eta(x) \int_{\mathbb{R}} u \, ds
$$

is linear and continuous since $|l(u)| \leq \|u\|_1 + |\int u \, ds| \leq 2\|u\|_1$. Furthermore if $u \in L^1([-m,m]; \mathbb{R})$ for some $m$ then $l(u)$ is compactly supported i.e. $\text{supp}(l(u)) \subset [-m,m]$.

This enables us to define a linear functional on $L^1([-m,m]; \mathbb{R})$ by

$$
\lambda(u) := \int_{\mathbb{R}} l(u) \, d\mu.
$$
The dual space of $L^1([-m,m];\mathbb{R})$ is isometrical isomorphic to $L^\infty([-m,m];\mathbb{R})$ so that for every $m \in \mathbb{N}$ there is a function $f_m \in L^\infty([-m,m];\mathbb{R})$ such that

$$\int_{\mathbb{R}} u f_m \, dx = \int_{\mathbb{R}} l(u) \, d\mu \quad \forall u \in L^1([-m,m];\mathbb{R}).$$

This implies that $\int u f_m \, dx = \int u f_{m+1} \, dx$ for all $u \in L^1([-m,m],\mathbb{R}) \subset L^1([-m-1,m+1],\mathbb{R})$; hence $f_{m+1} = f_m$ on $[-m,m]$ using the fundamental lemma of calculus of variations. Thus we can define a function $f \in L^\infty_{\text{loc}}(\mathbb{R},\mathbb{R})$ by

$$f(x) = f_m(x) \text{ for any } m \text{ with } |x| < m.$$ 

By construction $f$ has the property that for any $m \in \mathbb{N}$ and $u \in L^1([-m,m];\mathbb{R})$

$$\int_{\mathbb{R}} u f \, dx = \int_{\mathbb{R}} l(u) \, d\mu.$$ 

Observe that if $g \in C^1_c(\mathbb{R};\mathbb{R})$ we have $g' = \frac{dg}{dx} \in L^1([-m,m],\mathbb{R})$ for some $m$ large enough and $\int g' \, dx = 0$ so that $l(g')(x) = \int_{-\infty}^x g'(s) \, ds = g(x)$ and therefore

$$\int_{\mathbb{R}} g' f \, dx = \int_{\mathbb{R}} g \, d\mu. \quad \square$$

In $\mathbb{R}^n$ the situation is not that simple. Nonetheless the necessary condition is sufficient as summarised in the following lemma.

**Lemma 12.** Let $\mu$ be a positive Radon measure on $\mathbb{R}^n$ and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ $\mu$-measurable function with $|\sigma| = 1$ $\mu$-a.e., such that

$$\int g \cdot \sigma \, d\mu = 0 \text{ for all } g \in C^1_c(\mathbb{R}^n;\mathbb{R}^n) \text{ with } \text{div} \, g = 0 \quad (2.1)$$

Then there exists a function $f \in L^q_{\text{loc}}(\mathbb{R}^n;\mathbb{R})$ for all $1 \leq q < \frac{n}{n-1}$ such that

$$\int f \, \text{div} \, g \, dx = \int g \cdot \sigma \, d\mu \quad \forall g \in C^1_c(\mathbb{R}^n;\mathbb{R}^n). \quad (2.2)$$

$f$ is determined uniquely up to additive constants.

My first approach was to try to prove it in a similar way as we did it in the 1 dimensional case. The following two lemmas contain the necessary theory to do so.

**Lemma 13.** $-\Delta$ is a homeomorphism between $W^{2,p}_0(B(0,R);\mathbb{R})$ and $\{g \in L^p(B(0,R);\mathbb{R}) : \int g \, dx = 0 \text{ and } \int x^i g \, dx = 0 \text{ for } i = 1, \ldots, n\}$ for any $R > 0$.

**Lemma 14.** Let $\mu$ be a positive Radon measure on $\mathbb{R}^n$ and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ $\mu$-measurable function with $|\sigma| = 1$ $\mu$-a.e., then there exists a function $f \in L^q_{\text{loc}}(\mathbb{R}^n;\mathbb{R})$ for all $1 \leq q < \frac{n}{n-1}$.
\[
\frac{n}{n-1} \text{ such that }
- \int f \Delta \varphi \, dx = \int \nabla \varphi \cdot \sigma \, d\mu \quad \forall \varphi \in C_c^2(\mathbb{R}^n; \mathbb{R}).
\]  

(2.3)

\(f\) is determined uniquely up to harmonic functions in the sense of distributions i.e. \(u \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R})\) is harmonic in the sense of distributions if \(\int u \Delta \varphi = 0 \forall \varphi \in C_c^2(\mathbb{R}^n; \mathbb{R})\).

## 2.1 Approximation of Radon measures

A very important tool in the proof of theorem 6 will be the idea of approximating Radon measures by smooth functions. We consider interior approximation demonstrating the general idea without any technical difficulties and secondly global approximation. As mentioned in the introduction to this chapter we will as well use this method to give a second proof to lemma 12.

**Lemma 15** (Local approximation by smooth functions). Let \(\mu\) be a Radon measure on a open subset \(U \subset \mathbb{R}^n\). Let \(V \subset U\) and \(\delta > 0\) such that \(V_{2\delta} := \{x \in \mathbb{R}^n : \text{dist}(x, V) < 2\delta\}\) \(\subset U\). There is a sequence of functions \(\{f_\epsilon\}_{0 < \epsilon < \delta} \subset C_\infty(V; \mathbb{R})\) such that for any \(\varphi \in C_c(V; \mathbb{R})\)

\[
\lim_{\epsilon \to 0} \int f_\epsilon \varphi \, dx = \int \varphi \, d\mu \quad \forall \varphi \in C_c(V; \mathbb{R}) \text{ in fact }
\int f_\epsilon \varphi \, dx = \int \rho_\epsilon \ast \rho_\epsilon \ast \varphi \, d\mu.
\]  

(2.4)

Furthermore for any open subset \(A \subset V\) and \(0 < \epsilon < \delta\)

\[
\int_A |f_\epsilon| \, dx \leq |\mu|(A_{2\epsilon}) \quad \text{with } A_{2\epsilon} := \{x \in \mathbb{R}^n : \text{dist}(x, A) < 2\epsilon\}.
\]  

(2.5)

\(\rho_\epsilon\) denotes the standard mollifier.

**Remark 2.** Consider if \(\mu\) is a Radon measure on \(\mathbb{R}^n\) so that \(U = \mathbb{R}^n\) we can choose \(V\) equal to \(U\) and \(\delta > 0\) arbitrary; so that there are no restrictions on the functions and sets.

Giving up the particular nice form of equation (2.4) one can overcome the restrictions imposed by the smaller set \(V\).

**Lemma 16** (Global approximation by smooth functions). Let \(\mu\) be a Radon measure on an arbitrary open subset \(U \subset \mathbb{R}^n\). Then there is a sequence of linear operators \(L_\epsilon : C_c(U; \mathbb{R}) \to C_\infty(U; \mathbb{R})\), such that \(L_\epsilon(\varphi) \to \varphi\) uniformly on \(U\) as \(\epsilon \to 0\) for every \(\varphi\). Additionally there is a sequence \(\{f_\epsilon\}_{0 < \epsilon < 1} \subset C_\infty(U; \mathbb{R})\) such that for every \(\varphi \in C_c(U; \mathbb{R})\)

\[
\lim_{\epsilon \to 0} \int f_\epsilon \varphi \, dx = \int \varphi \, d\mu \quad \text{in fact } \int f_\epsilon \varphi \, dx = \int L_\epsilon(\varphi) \, d\mu.
\]
2.2 Poincaré’s inequality

Poincaré’s inequality will be frequently used. Although it is a classical inequality we state it, since we will use a slightly different version to the general one that applies to a ball.

The following version is due to Tartar, presented in his lecture notes [Tar07] chapter 10 and titled ”Poincaré’s inequality”. A proof can be found there as well.

**Lemma 17** (Poincaré’s inequality essentially part (vi) of Lemma 10.2 in [Tar07]). Let $U \subset \mathbb{R}^n$ be open and $X$ be a subspace of $W^{1,p}(U;\mathbb{R})$ or $BV(U;\mathbb{R})$ such that the injection of $X$ into $L^p(U;\mathbb{R})$ or $L^1(U;\mathbb{R})$ respectively is compact. Then Poincaré’s inequality holds on $X$ if and only if the constant function 1 does not belong to $X$.

**Remark 3.** If $X$ is a subset of $W^{1,p}(U;\mathbb{R})$ not containing 1 and $U \subset \mathbb{R}^n$ open with $\partial U$ continuous, then Poincaré’s inequality holds i.e. there is a constant $C > 0$ depending just on $p$ and the set $U$ such that

$$\|f\|_p \leq C \|\text{grad}(f)\|_p \quad \forall f \in X$$

because the injection of $W^{1,p}(U;\mathbb{R})$ into $L^p(U;\mathbb{R})$ is compact. This classic compactness theorem can be found for instance in [EG92] section 4.6 theorem 1 or [Tar07] chapter 18 lemma 18.4.

Similarly if $X$ is a subset of $BV(U;\mathbb{R})$ not containing 1 and $U \subset \mathbb{R}^n$ open with $\partial U$ continuous, then Poincaré’s inequality holds i.e. there is a constant $C > 0$ depending just on the set $U$ such that

$$\int_U |f| \, dx \leq \|Df\|(U) = \sup \left\{ \int_U f \text{div}(\varphi) \, dx : \varphi \in C_c^1(U;\mathbb{R}^n), |\varphi| \leq 1 \right\},$$

since a compactness theorem for BV functions holds, e.g. section 5.2.3 Theorem 4 in [EG92].

**Lemma 18** (Compact embedding for convex bounded subsets). If $U \subset \mathbb{R}^n$ is a bounded open convex set, then the injection of $W^{1,p}(U;\mathbb{R})$ into $L^p(U;\mathbb{R})$ is compact for $1 \leq p < \infty$.

**Remark 4.** A convex set has a Lipschitz continuous boundary so therefore this statement is already covered by remark 3. This proof shows the compact embedding without establishing first the regularity of the boundary. Nonetheless it is worth to notice that the key observation that if $B(c,r) \subset U$ so is $B((1-s)x + sc, sr) \subset U$ for all $x \in U$ is as well the key to prove the regularity of the boundary.

**Proof.** We will prove that if a sequence $u_n$ is bounded in $W^{1,p}(U;\mathbb{R})$ then there exists a subsequence $u_{n(m)}$ which converges strongly to $u_\infty \in L^p(u;\mathbb{R})$.

It is enough to show that for every $\alpha > 0$ one can write $u_n = v_n + w_n$ such that
\[ \|w_n\|_p \leq \alpha \] and that from \( v_n \) one can extract a converging subsequence \( v_{n(m)} \); hence
\[ \lim_{m,l \to \infty} \|u_{n(m)} - u_{n(l)}\|_p \leq 2\alpha. \] Starting from the selected subsequence, one repeats the argument with \( \alpha \) replaced by \( \frac{\alpha}{2} \). Repeating the argument inductively one obtains a Cauchy sequence in form of the diagonal subsequence.

Fix any \( c \in U \) and let \( r > 0 \) such that \( B(c, r) \subset U \). For \( 0 \leq s \leq 1 \)
\[ \Phi_s(x) := (1-s)x + sc \]
defines a smooth diffeomorphism. Observe that \( \Phi_s \) maps the open set \( U_{sr} := \{ x \in \mathbb{R}^n : \text{dist}(x, U) < sr \} \) into \( U \). Let \( z \in U_{sr} \) then \( z = x + sy \) with \( x \in U \) and \( |y| < r \) then \( \Phi_s(z) = (1-s)x + s(c + (1-s)y) \in U \) because \( |(1-s)y| < r \).

For every \( u \in W^{1,p}(U; \mathbb{R}) \) and any \( 0 \leq s \leq \frac{1}{2} \) we define \( u_s \in W^{1,p}(U_{sr}; \mathbb{R}) \) as \( u \circ \Phi_s \).

**Claim #1.** The following inequality holds:
\[ \|u_s\|_{W^{1,p}(U_{sr})} \leq (1-s)^{\frac{n}{p}} \|u\|_{W^{1,p}(U)} \quad (2.6) \]
There is a constant \( C > 0 \) just dependent on \( U, c, n \) such that
\[ \|u - u_s\|_p \leq Cs \|Du\|_p \quad \forall u \in W^{1,p}(U) \quad (2.7) \]

**Proof of Claim #1.** Since \( D\Phi_s = (1-s)id \) we have
\[ \int_{U_{sr}} |u_s|^p \, dx = (1-s)^{-n} \int_{\Phi_s(U_{sr})} |u|^p \, dy \leq (1-s)^{-n} \int_U |u|^p \, dx. \]
\[ Du_s(x) = (1-s)Du(\Phi_s(x)) \] and using in the same way the change of variables
\[ \int_{U_{sr}} |Du_s|^p \, dx = (1-s)^{-n+1} \int_{\Phi_s(U_{sr})} |Du|^p \, dy \leq (1-s)^{-n+1} \int_U |Du|^p \, dx. \]
This proves the first inequality.

One proves \((2.7)\) for all \( u \in C^\infty(U; \mathbb{R}) \) and then it extends to \( W^{1,p}(U; \mathbb{R}) \) by density. For any \( u \in C^\infty(U; \mathbb{R}) \) one has
\[ |u(x) - u_s(x)| = | - \int_0^s \frac{d}{dt} (u \circ \Phi_t) \, dt | = \int_0^s Du(\Phi_t(x)) \cdot (x-c) \, ds | \leq |x-c| \int_0^s |Du(\Phi_t(x))| \, dt. \]
Using Jensen’s inequality and the change of variable formula as before together with
\[ \Phi_s(U) \subset U \] one gets
\[
\|u - u_n\|_p \leq C \int_0^s \|Du \circ \Phi_t\|_p \, dt \leq C \int_0^s (1 - t)^{-n} \, dt \|Du\|_p \\
\leq C 2^n t \|Du\|_p.
\]

We used that \(|x - c| < C\) because \(U\) is bounded and \((1 - s) \geq \frac{1}{2}\). This proves the claim.

We are going to apply the decomposition argument twice. Firstly set \(w_n = u_n - u_n \circ \Phi_s\) with \(0 < s < \frac{1}{2}\) small enough such that \(\|w_n\| < \alpha \forall n\); possible because of (2.7). Thus \(v_n = u_n \circ \Phi\) build a bounded sequence in \(W^{1,p}(U, \mathbb{R})\). It remains to extract a subsequence that is converging in \(L^p(U, \mathbb{R})\).

Assuming now that the functions \(u_n\) are a bounded sequence on the larger set i.e. a bounded sequence in \(W^{1,p}(U_{sr}, \mathbb{R})\). One uses \(v_n = \rho_\epsilon * u_n\) restricted to \(\overline{U}\) for \(0 < \epsilon < sr\); so that we can apply the Arzelà-Ascoli theorem to the sequence \(v_n\), as they form a uniformly bounded sequence of equicontinuous functions on the compact set \(\overline{U}\) to obtain a converging subsequence. Furthermore for \(w_n = u_n - \rho_\epsilon * u_n\) the usual estimates give
\[
|u_n(x) - \rho_\epsilon * u_n(x)| \leq \left( \int \rho_\epsilon(y) \, dy \right)^{\frac{1-p}{p}} \left( \int \rho_\epsilon(y)(u_n(x) - u_n(x - y))^p \right)
\]
and
\[
\|u_n(\cdot) - u_n(\cdot - y)\|_{L^p(U)} \leq |y| \|Du_n\|_{L^p(U_{sr})}
\]
using that \(|y| \leq \epsilon < sr\) and with it \(x - y \in U_{sr} \forall x \in U\). In summary we conclude
\[
\|w_n\|_{L^p(U)} \leq \epsilon \|Du_n\|_{L^p(U_{sr})}.
\]

\[ \square \]

2.3 Proofs

We consider \(L^p(E; \mathbb{R})\) and similarly \(W_0^{m,p}(E; \mathbb{R})\) as subset of \(L^p(U; \mathbb{R})\) and \(W^{m,p}(U; \mathbb{R})\) respectively for all subsets \(E \subset U\). Since each function \(u\) can be extended by 0 outside of \(E\) and gives a function \(\tilde{u} \in L(U; \mathbb{R})\) or \(W^{m,p}(U; \mathbb{R})\) respectively. We can extend the functions \(u \in W_0^{m,p}\) in that way since \(W_0^{m,p}(E; \mathbb{R})\) is by definition the closure of \(C_0^\infty(E; \mathbb{R})\) in \(W^{m,p}\).

**Proof of Lemma 13** Let \(R > 0\) be fixed.

To see that the image of the first set under \(-\Delta\) is indeed a subset of the second observe for \(u \in W^{1,p}(B(0,R); \mathbb{R})\) we do have integration by parts on this set, because \(\partial B(0,R)\) is nice. Hence
\[
- \int_{B(0,R)} \Delta u \, dx = - \int_{\partial B(0,R)} \nabla u \cdot \frac{x}{|x|} \, d\mathcal{H}^{n-1} = 0
\]
because \(\nabla u \in W_0^{1,p}(B(0,R); \mathbb{R}^n)\) and so has vanishing trace. Similarly
\[
- \int_{B(0,R)} x^i \Delta u \, dx =
\]
therefore it still holds that for each $m$ we have proved the lemma on the base of claim #1. The previous calculation reveals that $(C_{\infty} \text{element of } \text{function of } n \text{ \complex variables} F)$. Since all operators involved are linear and $C_{\infty} \text{element of } \text{function of } n \text{ \complex variables} F = \text{the solution of } -\Delta w = 0$ in $B(0, R)$ with Dirichlet’s boundary conditions, uniqueness for this PDE under the given boundary conditions provides that $w \equiv 0$ and therefore $u = v$.

Claim #1. Let $f \in C_{\infty}^\infty(B(0, R); \mathbb{R})$ with $\int f \, dx = 0$ and $\int x^i f \, dx = 0$ for $i = 1, \ldots, n$, then there is $u \in C_{\infty}^\infty(B(0, R); \mathbb{R})$ such that $-\Delta u = f$.

Proof of Claim #1. $\mathcal{F} f$ denotes the Fourier transform of $f$ and $\mathcal{F} f$ its inverse. Then $\mathcal{F} f(0) = \frac{1}{(2\pi)^{n/2}} \int f \, dx = 0$ and $\frac{\partial f}{\partial \xi_j}(0) = \frac{1}{(2\pi)^{n/2}} \int (-ix_j) f \, dx = 0$ for $j = 1, \ldots, n$. The Paley-Wiener theorem (compare e.g. [RS75]) provides that $\mathcal{F} f$ is an entire analytic function of $n$ complex variables $\xi = (\xi_1, \ldots, \xi_n)$ and for each $m \in \mathbb{N}$ there is a constant $C_m$ so that

$$|\mathcal{F} f(\xi)| \leq \frac{C_m e^{R|\text{Im}(\xi)|}}{(1 + |\xi|)^m}.$$ 

The previous calculation reveals that $\xi^{-2} \mathcal{F} f(\xi)$ can be extended analytically in $0$ and therefore it still holds that for each $m \in \mathbb{N}$ there is a perhaps changed constant $C_m$ so that

$$|\xi^{-2} \mathcal{F} f(\xi)| \leq \frac{C_m e^{R|\text{Im}(\xi)|}}{(1 + |\xi|)^m}.$$ 

Using the ”if” part of the Paley-Wiener theorem we obtain that $u := \mathcal{F}^{-1}(\xi^{-2} \mathcal{F} f)$ is an element of $C_{\infty}^\infty(B(0, R); \mathbb{R})$. By construction $u$ solves $-\Delta u = f$ that proves the claim.

Observe that $-\Delta u = f$ on whole $\mathbb{R}^n$ with $u, f$ considered as functions in $C_{\infty}^\infty(\mathbb{R}^n; \mathbb{R})$. Hence $u$ coincide with the fundamental solution i.e. the convolution with the Newton potential. The Calederon-Zygmund inequality, compare Theorem 9.9 and Corollary 9.10 in [GT01] states that there is a constant $C$ just depending on $n, p, R$ such that for all $g \in L^p(B(0, R); \mathbb{R})$ and $v$ the solution of $-\Delta v = g$ obtained by convolution with the Newton potential.

$$\int_{\mathbb{R}^n} |D^2 v|^p \, dx \leq C \int_{B(0, R)} |g|^p \, dx.$$ 

Since all operators involved are linear and $C_{\infty}^\infty(B(0, R); \mathbb{R}) \cap \{ \int g \, dx = 0, \int x^i g \, dx = 0 \ (i = 1, \ldots, n) \}$ is dense in $\{ g \in L^p(B(0, R); \mathbb{R}) : \int g \, dx = 0, \int x^i g \, dx = 0 \ (i = 1, \ldots, n) \}$ we have proved the lemma on the base of claim #1. \hfill \Box

Proof of Lemma [74] The uniqueness statement follows from the linearity of (2.3). Let
If \( f, \tilde{f} \) be admissible solutions then \( w := f - \tilde{f} \) satisfies
\[
\int w \Delta \varphi \, dx = 0 \quad \forall \varphi \in C_c^2(\mathbb{R}^n; \mathbb{R})
\]
thus \( w \) is weakly harmonic.

To prove the existence we fix functions \( \eta, \eta_i \in C^\infty(\mathbb{R}^n, \mathbb{R}) \) for \( i = 1, \ldots, n \) with the following properties

1. \( \text{supp}(\eta), \text{supp}(\eta_i) \subset \overline{B}(0, 1) \),
2. \( \int \eta \, dx = 1 \) and \( \int x_j \eta \, dx = 0 \) (\( j = 1, \ldots, n \)),
3. \( \int \eta_i \, dx = 0 \) and \( \int x_j \eta_i \, dx = \delta_{ij} \) for \( i, j = 1, \ldots, n \).

A possible set of function would be
\[
\eta(x) := \begin{cases} 
  c \exp\left(\frac{-1}{1-|x|^2}\right) & \text{if } |x| < 1 \\
  0 & \text{if } |x| > 1
\end{cases}
\]
for an appropriate constant \( c \) such that \( \int \eta \, dx = 1 \). It is smooth, symmetric and positive

furthermore \( \text{supp}(\eta) \subset \overline{B}(0, 1) \). It is straight forward to check that \( \eta_i(x) := -\frac{\partial \eta}{\partial x_i}(x) \) has the desired properties.

\[
m(g) := \int g \, dx \\
m_i(g) := \int x_i g \, dx \quad \text{for } i = 1, \ldots, n
\]
are bounded linear functionals on \( L^p(B(0, R); \mathbb{R}) \) since \( |m(g)| \leq L^p(B(0, R))^{1-1/p} \|g\|_p \)
and \( |m_i(g)| \leq R L^p(B(0, R))^{1-1/p} \|g\|_p \). Thus
\[
L(g) := g - m(g) \eta - \sum_{i=1}^{n} m_i(g) \eta_i
\]
maps \( L^p(B(0, R); \mathbb{R}) \) continuously onto \( \{g \in L^p(B(0, R); \mathbb{R}) : \int g \, dx = 0, \int x_i g \, dx = 0 \ (i = 1, \ldots, n)\} \) for every \( R > 1 \).

For every fixed \( R > 1 \) lemma 13 implies that \( A := (-\Delta)^{-1} \) is a bounded linear operator mapping \( \{g \in L^p(B(0, R); \mathbb{R}) : \int g \, dx = 0, \int x_i g \, dx = 0 \ (i = 1, \ldots, n)\} \) onto \( W_0^{2,p}(B(0, R); \mathbb{R}) \). Fix any \( p \) with \( n < p < \infty \) so that \( W_0^{2,p}(B(0, R); \mathbb{R}) \) is continuously embedded into \( C_c^1(B(0, R); \mathbb{R}) \) by Sobolev’s embedding theorem, compare e.g. Theorem 6.7 in [Tar07]. Thus \( A \circ L : L^p(B(0, R); \mathbb{R}) \to C^1(\mathbb{R}^n; \mathbb{R}) \cap \{\text{supp}(g) \subset \overline{B}(0, R)\} \) is continuous.
For every \( R > 1 \) the following defines a linear functional on \( L^p(B(0, R); \mathbb{R}) \)
\[
\lambda(g) := \int \nabla((A \circ L)(g)) \cdot \sigma \, d\mu;
\]
linearity is obvious, continuity follows by the construction above and since \( \mu \) is a Radon measure the map \( h \in C_c(\mathbb{R}^n; \mathbb{R}) \mapsto \int h \cdot \sigma \, d\mu \) is continuous. The dual space of \( L^p(B(0, R); \mathbb{R}) \) is isometrical isomorphic to \( L^q(B(0, R); \mathbb{R}) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) such that for every \( R > 1 \) there is \( f_R \in L^q(B(0, R); \mathbb{R}) \) such that \( \int g f_R \, dx = \lambda(g) \) for all \( g \in L^p(B(0, R); \mathbb{R}) \). \( p \in (n, \infty) \) has been arbitrary and \( L^q(B(0, R); \mathbb{R}) \subset L^p(B(0, R); \mathbb{R}) \) for \( \tilde{p} \geq p \) therefore \( f_R \in L^q(B(0, R); \mathbb{R}) \) for all \( 1 < q < \frac{n}{n-1} \).

Furthermore \( L^p(B(0, R); \mathbb{R}) \subset L^p(B(0, R'); \mathbb{R}) \) for \( R \leq R' \) so that
\[
\int g f_R \, dx = \int \nabla((A \circ L)(g)) \cdot \sigma \, d\mu = \int g f_{R'} \, dx \quad \forall g \in L^p(B(0, R); \mathbb{R}).
\]
Applying the fundamental lemma of calculus of variation we obtain that \( f_R \equiv f_{R'} \) on \( B(0,R) \), so that we define \( f \in L^q_{\text{loc}}(\mathbb{R}^n, \mathbb{R}) \) and \( 1 \leq q < \frac{n}{n-1} \) by
\[
f(x) = f_R(x) \text{ for any } R > |x|.
\]

Let \( \psi \in C^2_c(\mathbb{R}^n; \mathbb{R}) \) set \( g = -\Delta \psi \). By simple application of the divergence theorem one obtains \( \int g \, dx = -\int \Delta \psi = 0 \) and \( \int x_i g \, dx = -\int x_i \Delta \psi = 0 \) for \( i = 1, \ldots, n \). Hence \( L(-\Delta \psi) = -\Delta \psi \) and \( (A \circ L)(-\Delta \psi) = A(-\Delta \psi) = \psi \) in summary
\[
-\int \Delta \psi f \, dx = \int \nabla \psi \cdot \sigma \, d\mu \quad \forall \psi \in C^\infty_c(\mathbb{R}^n; \mathbb{R}).
\]

After we have proved these two technical lemmas we can prove finally the lemma of interest.

**Proof of lemma** Again we start with the ”uniqueness” part. Suppose \( f, \tilde{f} \) satisfying (2.2), thus \( w = f - \tilde{f} \in L^q_{\text{loc}}(\mathbb{R}^n, \mathbb{R}) \) for \( 1 \leq q < \frac{n}{n-1} \) satisfies
\[
\int w \text{ div } g \, dx = 0 \quad \forall g \in C^1_0(\mathbb{R}^n; \mathbb{R}^n).
\]
Let \( \rho_\epsilon \) denote the standard mollifier then
\[
\int \rho_\epsilon \ast w \text{ div } \varphi \, dx = \int w \text{ div}(\rho_\epsilon \ast \varphi) \, dx = 0 \quad \forall \varphi \in C^1_0(\mathbb{R}^n; \mathbb{R}^n).
\]
But $\rho \ast w \to w$ in $L^q_{\text{loc}}$ and $\rho \ast w \in C^\infty(\mathbb{R}^n, \mathbb{R})$. Integration by parts reveals

$$ \int \nabla(\rho \ast w) \cdot \varphi \, dx = 0 \quad \forall \varphi \in C^1_c(\mathbb{R}^n; \mathbb{R}). $$

Thus $\nabla(\rho \ast w) = 0$ or $\rho \ast w = \text{const.}$ for every $\epsilon > 0$. But since $\rho \ast w$ converges to $w$ in $L^q_{\text{loc}}$, $w$ has to be constant almost everywhere. This proves the uniqueness statement.

It remains to prove existence of such an $f$. As before fix $\eta \in C^\infty(\mathbb{R}^n; \mathbb{R})$ such that $\supp(\eta) \subset B(0,1)$ and $\int \eta \, dx = 1$.

Claim #1. For any $g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ there is $w \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$, $q \in C^2_c(\mathbb{R}^n; \mathbb{R})$ such that with

$$ m(g) := \int g \, dx = (\int g_1 \, dx, \ldots, \int g_n \, dx) \in \mathbb{R}^n $$

1. $\text{div}(w) = 0$;
2. $g = w - \nabla q + \eta m(g)$ so that $\text{div}(g) = -\Delta q + \nabla \eta \cdot m(g)$.

Proof of Claim #1. For $g = (g_1, \ldots, g_n) \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ set $h := g - \eta m(g)$. Firstly observe that $\int \text{div}(h) \, dx = 0$ and $\int x_i \text{div}(h) \, dx = -\int h_i \, dx = 0$ for $i = 1, \ldots, n$ by a straight forward application of the divergence theorem using $h$ compactly supported and its particular definition. Applying lemma 13 to $\text{div}(h)$ there is $q \in C^2_c(\mathbb{R}^n; \mathbb{R})$ such that $-\Delta q = \text{div}(h)$. With $w = h + \nabla q$ an element in $C^1_c(\mathbb{R}^n; \mathbb{R})$ and $\text{div}(w) = 0$, we have proved the claim since $g = h + \nabla q + \eta m(g) = w - \nabla q + \eta m(g)$.

For any $a \in \mathbb{R}^n$ and $g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ we have

$$ \int (a \cdot x) \, \text{div}(g) \, dx = -a \cdot m(g). \quad (2.8) $$

This follows directly from the divergence theorem and $\nabla(b \cdot x) = b$.

Claim #2. Let $\tilde{f}$ be any function in $L^q_{\text{loc}}(\mathbb{R}^n; \mathbb{R})$ of lemma 14 i.e. $\tilde{f}$ satisfies (2.3). Then the function $f \in L^q_{\text{loc}}(\mathbb{R}^n; \mathbb{R})$ defined as follows has the desired properties i.e. it satisfies (2.2).

$$ f(x) = -\tilde{f}(x) - (b + c) \cdot x $$

with the vectors

$$ b = \int \eta \sigma \, d\mu \in \mathbb{R}^n $$

$$ c = \int \nabla \eta \tilde{f} \, dx \in \mathbb{R}^n. $$

Proof of Claim #2. Let $g \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ be arbitrary, and $w,q$ the function out of claim #1. Then applying (2.8) several times and the defining properties of $w,q$ and $\tilde{f}$
we calculate
\[ \int f \text{div}(g) \, dx = - \int \tilde{f} \text{div}(g) \, dx - \int ((b + c) \cdot x) \text{div}(g) \, dx \]
\[ = - \int \tilde{f} (-\Delta q + \nabla \eta \cdot m(g)) \, dx + (b + c) \cdot m(g) \]
\[ = - \int \nabla q \cdot \sigma \, d\mu + m(g) \cdot b - m(g) \cdot \left( \int \tilde{f} \nabla \eta \, dx \right) + c \cdot m(g) \]
\[ = \int (-\nabla q) \cdot \sigma \, d\mu + \int (\eta m(g)) \cdot \sigma \, d\mu + \int w \cdot \sigma \, d\mu \]
\[ = \int g \cdot \sigma \, d\mu; \]
proving the claim and the lemma. \qed

In the following we present the proof to the for us important lemma \ref{lem:approx}, the local and the global approximation variant.

**Proof of lemma** \ref{lem:approx}. Fix a sequence of compact set \( K_m \subset V_\delta = \{ x \in \mathbb{R}^n : \text{dist}(x,V) < \delta \} \) with \( K_m \subset K_{m+1} \) and \( \bigcup_m K_m = V_\delta \). For example \( K_m = B(0,\epsilon) \cap \overline{V_\delta - \frac{1}{m}} \). By construction all these are subsets of \( U \). Furthermore if \( K \) is any compact subset of \( V_\delta \) and \( 0 < \epsilon < \delta \) so is \( K_\epsilon := \{ x \in \mathbb{R}^N : \text{dist}(x,K) \leq \epsilon \} \) still a compact subset of \( U \).

Let \( 0 < \epsilon < \delta \) be arbitrary. \( \rho_\epsilon \) denotes the standard mollifier. For every \( m \in \mathbb{N} \) set \( K_m^\epsilon \) as the compact subset of \( U \) given by \( \{ x \in \mathbb{R}^n : \text{dist}(x,K_m) \leq \epsilon \} \).

Observe that if \( \varphi \in L^1(K_m;\mathbb{R}) \) considered as an element of \( L^1(U;\mathbb{R}) \) extending it by 0 \( \rho_\epsilon \ast \varphi \in C_0^\infty(U;\mathbb{R}) \). Actually \( \text{supp}(\rho_\epsilon \ast \varphi) \subset K_m^\epsilon \). And \( |\rho_\epsilon \ast \varphi| \leq ||\varphi||_1 \). Hence the map from \( L^1(K_m;\mathbb{R}) \) into \( C_0^\infty(U;\mathbb{R}) \cap \{ \text{supp}(\psi) \subset K_m^\epsilon \} \)
\[ \varphi \mapsto \rho_\epsilon \ast \varphi \]
is continuous and therefore
\[ \lambda_\epsilon(\varphi) := \int \rho_\epsilon \ast \varphi \, d\mu \]
defines a linear functional on \( L^1(K_m;\mathbb{R}) \) for every \( m \) and \( 0 < \epsilon < \delta \). The dual space to \( L^1(K_m;\mathbb{R}) \) is isometric isomorphic to \( L^\infty(K_m;\mathbb{R}) \). Thus for every \( m \in \mathbb{N} \), \( 0 < \epsilon < \delta \) there is a function \( f_m^\epsilon \in L^\infty(K_m;\mathbb{R}) \) such that \( \int \varphi f_m^\epsilon \, dx = \int \rho_\epsilon \ast \varphi \, d\mu \) for all \( \varphi \in L^1(K_m;\mathbb{R}) \).

As before \( \int \varphi f_m^\epsilon \, dx = \int \varphi f_{m+1}^\epsilon \, dx \forall \varphi \in L^1(K_m;\mathbb{R}) \) since \( L^1(K_m;\mathbb{R}) \subset L^1(K_{m+1};\mathbb{R}) \) for all \( m \) and therefore we can define \( f_\epsilon \in L^\infty_{\text{loc}}(V_\delta) \) as follows
\[ f_\epsilon(x) := f_m^\epsilon(x) \text{ for any } m \text{ large enough such that } x \in K_m. \]

Finally the function \( f_\epsilon \), an element \( C^\infty(V;\mathbb{R}) \), given by
\[ f_\epsilon = \rho_\epsilon \ast f_\epsilon. \]
It is well defined on $V$ since there is enough space to smooth because $B(x, \epsilon) \subset V_{\delta}$ for all $x \in V, 0 < \epsilon < \delta$. $f_\epsilon$ has the desired properties since

$$\int \varphi f_\epsilon \, dx = \int \rho_\epsilon \ast \varphi f_\epsilon \, dx = \int \rho_\epsilon \ast \rho_\epsilon \ast \varphi \, d\mu \quad \forall \varphi \in L^1(V; \mathbb{R}).$$

$\rho_\epsilon \ast \rho_\epsilon \ast \varphi \to \varphi$ as $\epsilon \to 0$ uniformly on $V$ for $\varphi \in C_c(V; \mathbb{R})$ because

$$|\rho_\epsilon \ast \rho_\epsilon \ast \varphi(x) - \varphi(x)| = \left| \int \rho_\epsilon(y) \rho_\epsilon(z) (\varphi(x-y-z) - \varphi(x)) \right| \leq \sup \{ |\varphi(x) - \varphi(y)| : |x-y| \leq 2\epsilon \}.$$

This implies the convergence.

It remains to prove the bound (2.5). Let $W \subset V$ be open then

$$\int_W |f_\epsilon| \, dx = \sup \{ \int f_\epsilon \varphi \, dx : \varphi \in C^\infty(U; \mathbb{R}), \text{supp}(\varphi) \subset W, |\varphi| \leq 1 \}$$

$$= \sup \{ \int \rho_\epsilon \ast \rho_\epsilon \ast \varphi \, d\mu : \varphi \in C^\infty(U; \mathbb{R}), \text{supp}(\varphi) \subset W, |\varphi| \leq 1 \}$$

$$\leq |\mu|(W_2\epsilon)$$

since $|\rho_\epsilon \ast \rho_\epsilon \ast \varphi| \leq ||\varphi||_{\infty} \leq 1$ and $\text{supp}(\rho_\epsilon \ast \rho_\epsilon \ast \varphi) \subset W_2\epsilon$.

To prove the global approximation the general idea remains. Now one has to be careful with the space needed to smooth close to the boundary, this is overcome by constructing an appropriate smoothing operator $L_\epsilon$.

**Proof of the lemma** 16 Fix some $k_0 \in \mathbb{N}$ arbitrary large and define the open sets

$$U_k := \{ x \in U : \text{dist}(x, \partial U) > \frac{1}{k_0 + k} \}.$$

With $U_0 := \emptyset$ set

$$V_k := U_{k+1} - \overline{U}_{k-1}.$$

Consider that by construction $V_{k+2} \cap V_k = \emptyset$ for all $k \in \mathbb{N}$ and $\bigcup_{k \in \mathbb{N}} V_k = U$. By choosing $\epsilon_k := \frac{k_0 + k + 1}{(k_0 + k + 1)!}$ we achieve that $(V_k)_{\epsilon_k} \subset V_{k-1} \cup V_k \cup V_{k+1}$ for all $k \in \mathbb{N}$ using the notation $A_\delta := \{ x \in \mathbb{R}^n : \text{dist}(x, A) < \delta \}$.

Let $\{ \eta_k \}_{k \in \mathbb{N}}$ be a smooth partition of unity associated to $\{ V_k \}_{k}$. Consider that by construction $\rho_\epsilon \ast (\eta_k \varphi) \in C^\infty_c(\bigcup_{k=1}^{k+1} V; \mathbb{R})$ for all $k$. Define for $\varphi \in L^1_{\text{loc}}(U; \mathbb{R})$

$$l_\epsilon(\varphi) := \sum_{k=1}^{\infty} \rho_\epsilon \ast (\eta_k \varphi).$$

In some neighbourhood of each point $x \in U$ there are at most finitely many terms in that sum nonzero; therefore $l_\epsilon(\varphi) \in C^\infty(U; \mathbb{R})$. $\varphi = \varphi^+ - \varphi^-$ with $\varphi^+ = \max(\varphi, 0)$ and $\varphi^-$ = ...
\[ \varphi^- = \max(-\varphi, 0) \text{ and } \rho_{\epsilon_k}, \eta_k \geq 0, \text{ thus } l_\epsilon(\varphi) = l_\epsilon(\varphi^+) - l_\epsilon(\varphi^-) \] 

\[ -l_\epsilon(\varphi^+) + l_\epsilon(\varphi^-) \leq l_\epsilon(\varphi) \leq l_\epsilon(\varphi^+) + l_\epsilon(\varphi^-), \] 

but \( |\varphi| = \varphi^+ + \varphi^- \) and \( \rho_{\epsilon_k} \ast (\eta_k |\varphi|) \leq \int \eta_k |\varphi| \, dx \); hence 

\[ |l_\epsilon(\varphi)| \leq l_\epsilon(\varphi^+) + l_\epsilon(\varphi^-) = \sum_k \rho_{\epsilon_k} \ast (\eta_k |\varphi|) \leq \sum_k \int \eta_k |\varphi| \, dx = \int |\varphi| \, dx. \] 

This shows that we constructed a continuous linear operator 

\[ l_\epsilon : L^1_{\text{loc}}(U; \mathbb{R}) \to C^\infty(U; \mathbb{R}) \]

\[ \varphi \mapsto \sum_{k=1}^\infty \rho_{\epsilon_k} \ast (\eta_k \varphi). \]

Consider the sequence of compact subsets of \( U \) 

\[ K_m := \overline{U_{m+1} \cap B(0, k_0 + m)}; \]

then \( l_\epsilon \) has the further nice property that \( \text{supp}(l_\epsilon(\varphi)) \subset K_{m+1} \) whenever \( \varphi \in L^1(K_m; \mathbb{R}) \) extended by 0 outside of \( K_m \).

As before we use the continuous linear functional 

\[ \lambda_\epsilon(\varphi) = \int l_\epsilon(\varphi) \, d\mu \]

and the isomorphism between the dual space of \( L^1(K_m; \mathbb{R}) \) and \( L^\infty(K_m; \mathbb{R}) \) to obtain functions \( f^\epsilon_m \in L^\infty(K_m; \mathbb{R}) \) satisfying 

\[ \int \varphi f^\epsilon_m \, dx = \int l_\epsilon(\varphi) \, d\mu \quad \forall \varphi \in L^1(K_m; \mathbb{R}). \]

As before we can define now \( \tilde{f}_\epsilon \in L^\infty_{\text{loc}}(U; \mathbb{R}) \) as 

\[ \tilde{f}_\epsilon(x) := f^\epsilon_m(x) \text{ for any } m \text{ large enough such that } x \in K_m \]

and \( f_\epsilon := l_\epsilon(\tilde{f}_\epsilon) \in C^\infty(U; \mathbb{R}). \)

Observe that \( \eta_k(\rho_{\epsilon_k} \psi) \) is well defined for any \( \psi \in L^1_{\text{loc}}(U; \mathbb{R}) \) since for any \( x \in V_k \) \( \overline{B(x, \epsilon_k)} \subset U \); hence the transposed operator \( l^*_\epsilon \) given as 

\[ l^*_\epsilon(\varphi) = \sum_{k=1}^\infty \eta_k(\rho_{\epsilon_k} \ast \varphi) \]

maps elements of \( L^1_{\text{loc}}(U; \mathbb{R}) \) into \( C^\infty(U; \mathbb{R}) \) because to every \( x \in U \) there is a neighbourhood such that at most 3 terms of this sum are nonzero. Actually if \( x \in V_k, V_k \)
Writing out the convolution taking into account that
\[ \eta \rho \]
therefore
Furthermore observe that if \( x \psi \)
We have seen that for any
This is possible since
\{ k, l \}
\supp
and therefore
\[ \int \left| \eta_k(x+y)(x-y-z) - \eta_k(x)\eta_l(x)\varphi(x) \right| < \frac{\alpha}{15} \] for all \( x \in U \).

Let \( x \in supp(L_\epsilon(\varphi)) \) and \( m \leq M - 3 \) such that \( x \in V_m \). Using \( \sum_k \eta_k = 1 \) on \( U \) we calculate
\[
\left| L_\epsilon(\varphi)(x) - \varphi(x) \right| = \left| \sum_{k,l} (\rho_{\epsilon_k} * (\eta_k \eta_l \varphi))(x) - \eta_k(x)\eta_l(x)\varphi(x) \right|
\]
\[
= \left| \sum_{k=m-2}^{m+2} \sum_{l=k-1}^{k+1} (\rho_{\epsilon_k} * (\eta_k \eta_l \varphi))(x) - \eta_k(x)\eta_l(x)\varphi(x) \right|
\].

Writing out the convolution taking into account that \( \iint \rho_{\epsilon_k}(y)\rho_{\epsilon_l}(y-z) \, dz \, dy = 1 \), we
obtain for $\epsilon < \delta$ and so $\epsilon_k < \delta$

$$\left| L_\epsilon(\varphi)(x) - \varphi(x) \right|$$

$$= \left| \sum_{k=m-2}^{m+2} \sum_{l=k-1}^{k+1} \int \int \rho_{\epsilon_k}(y)\rho_{\epsilon_l}(z-y) \left( \eta_k(x-y)\eta_l(x-y)\varphi(x-y-z) - \eta_k(x)\eta_l(x)\varphi(x) \right) \, dz \, dy \right|$$

$$\leq \sum_{k=m-2}^{m+2} \sum_{l=k-1}^{k+1} \sup \left\{ \left| \eta_k(u-y)\eta_l(u-y)\varphi(u-y-z) - \eta_k(u)\eta_l(u)\varphi(u) \right| : u \in U, |y|, |z| \leq \delta \right\}$$

$$< \alpha.$$ 

This proves the uniform convergence. \(\square\)

As mentioned in the introduction we present now a proof of lemma 12 exploiting the approximation by smooth functions. In fact we can improve the statement slightly by replacing $\mathbb{R}^n$ by arbitrary open convex subsets $U$:

**Lemma 19** (Lemma [12] revised). Let $U$ be an open convex subset of $\mathbb{R}^n$, $\mu$ be a non-negative Radon measure on $U$ and $\sigma : U \to \mathbb{R}^n$ a $\mu$-measurable function with $|\sigma| = 1$ $\mu$-a.e., such that

$$\int g \cdot \sigma \, d\mu = 0 \quad \forall g \in C^1_c(U; \mathbb{R}^n) \text{ with } \text{div } g = 0$$

Then there exists a function $f \in BV_{\text{loc}}(U; \mathbb{R}^n)$ such that $[Df] = -\sigma \mu$. $f$ is determined uniquely up to additive constants.

**Proof.** The uniqueness is proven as before.

Choose a sequence of non empty open subsets $V_m \subset U$ with the following properties

1. for every $m$ $V_m$ is bounded open convex subset;
2. $V_m \subset V_{m+1}$ for every $m$ and $\bigcup_{m=1}^{\infty} V_m = U$;
3. for every $m$ there is $\delta_m > 0$ s.t. $(V_m)_{2\delta_m} = \{x \in \mathbb{R}^n : \text{dist}(x, V_m) < 2\delta_m \} \subset U$.

Note that this implies that $\overline{V}_m$ and $(\overline{V}_m)_{\delta_m} = \{x \in \mathbb{R}^n : \text{dist}(x, V_m) \leq \delta_m \}$ are compact subset of $U$.

An admissible sequence would be $V_m := \{x \in U : \text{dist}(x, U) > \frac{3}{m} \} \cap B(0, m)$ together with $\delta_m = \frac{1}{m}$.

Fix $\eta \in C^\infty_c(V_1; \mathbb{R})$ nonnegative with $\int \eta \, dx = 1$. 29
Consider a fixed $m$.

The particular choice of $V_m$ enables us to use lemma [15] to approximate the Radon measures on $V_m$; hence for every $i = 1, \ldots, n$ there is a sequence $\{F^\epsilon_i\}_{0<\epsilon<\delta_m} \subset C^\infty(V_m; \mathbb{R})$ such that with $\sigma = (\sigma_1, \ldots, \sigma_n)$

$$
\int \varphi \, F^\epsilon_i \, dx = \int (\rho_\epsilon \ast \rho_\epsilon \ast \varphi) \, \sigma_i \, d\mu \quad \forall \varphi \in C_c(V_m; \mathbb{R}) \ 0<\epsilon<\delta_m.
$$

Claim #1. Every $F^\epsilon := (F^\epsilon_1, \ldots, F^\epsilon_n) \in C^\infty(V_m; \mathbb{R})$ is a closed covector field.

Proof of Claim #1. Firstly we observe that $\text{div}(\rho_\epsilon \ast \rho_\epsilon \ast g) = \rho_\epsilon \ast \rho_\epsilon \ast \text{div}(g)$. Therefore (2.1) is preserved i.e.:

$$
\int g \cdot F^\epsilon \, dx = 0 \quad \text{for all } g \in C^1_c(V_m; \mathbb{R}^n) \text{ with } \text{div} \ g = 0. \quad (2.10)
$$

We can choose for any $\varphi \in C^\infty_c(V_m; \mathbb{R})$

$$
g = (\ldots, 0 \cdot \frac{\partial \varphi}{\partial x^i}, \ldots, 0 \cdot \frac{\partial \varphi}{\partial x^j}, \ldots).
$$

Thus $\text{div} \ g = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} - \frac{\partial^2 \varphi}{\partial x^j \partial x^i} = 0$. All involved functions are smooth and we can integrate by parts in (2.10) and we obtain for this particular choice of $g$

$$
0 = \int g \cdot F^\epsilon \, dx = \int \frac{\partial \varphi}{\partial x^i} F^\epsilon_i - \frac{\partial \varphi}{\partial x^j} F^\epsilon_j \, dx
= - \int \varphi \left( \frac{\partial F^\epsilon_i}{\partial x^i} - \frac{\partial F^\epsilon_j}{\partial x^j} \right) \, dx.
$$

$\varphi$ has been arbitrary; hence $\frac{\partial F^\epsilon_i}{\partial x^i} - \frac{\partial F^\epsilon_j}{\partial x^j} \equiv 0$ for $i, j = 1, \ldots, n$ on $V_m$, proving that $F^\epsilon$ is closed.

Claim #2. For every $m \in \mathbb{N}$ there is $f_m \in BV(V_m; \mathbb{R})$ satisfying (2.2) on $V_m$ i.e. 

$$
\int f_m \text{div} \ g \, dx = \int g \cdot \sigma \, d\mu \quad \forall g \in C^1_c(V_m; \mathbb{R}^n).
$$

Proof of Claim #2. It is enough to consider a fixed $m$. $V_m$ is a star-shaped open subset of $\mathbb{R}^n$, then closeness implies exactness on $V_m$ ([Lee03] Proposition 6.30). Thus there is $f_\epsilon \in C^\infty(V_m; \mathbb{R})$ such that $Df^\epsilon_m = F^\epsilon_i$. Choosing the constant of integration appropriate we can ensure that $\int \eta f^\epsilon_m \, dx = 0$ for every $0<\epsilon<\delta_m$.

Poincaré’s inequality holds on the set $\{u \in W^{1,1}(V_m; \mathbb{R}) : \int \eta u \, dx = 0\}$. This is a consequence of the particular choice of $V_m$ and combining lemma [17] with lemma [18].
For $0 < \epsilon < \frac{\delta_m}{2}$. We can now estimate the $L^1$-norm of $f_\epsilon$ using (2.5)

$$\int_{V_m} |f_\epsilon| \, dx \leq C \int_{V_m} |Df_\epsilon| \, dx$$

$$= C \int_{V_m} |F_\epsilon| \, dx \leq C \sum_{i=1}^n |\mu_i| ((V_m)_{2\epsilon})$$

$$\leq C \sum_{i=1}^n |\mu_i| ((V_m)_{\delta_m}) < \infty.$$  

We can apply the compactness result of lemma [18] to the sequence $\{f_\epsilon\}_{0 < \epsilon < \frac{\delta}{2}}$ as they form a bounded sequence in $W^{1,1}(V_m; \mathbb{R})$ to extract a subsequence $f_{\epsilon_k}$ converging to a function $-f_m$ in $L^1(V_m; \mathbb{R})$. The lower semicontinuity of variation measure, compare [EG92] section 5.2.1 Theorem 1, implies that $f_m$ is itself an element of $BV(V_m; \mathbb{R})$ with $\|Df_m\|(V_m) = \limsup_{k \to \infty} -Df_{\epsilon_k}(V_m)$. Furthermore $f_m$ still satisfies

$$\int f_m \eta \, dx = -\lim_{k \to \infty} \int f_{\epsilon_k} \eta \, dx = 0$$

and more important for any $g \in C^1_c(V_m; \mathbb{R}^n)$

$$\int f_m \text{div}(g) \, dx = -\lim_{k \to \infty} \int \text{div}(g) f_{\epsilon_k} \, dx$$

$$= -\lim_{k \to \infty} \int g \cdot F_{\epsilon_k} \, dx$$

$$= \int g \cdot \sigma \, d\mu.$$  

**Claim #3.** Let for every $m$ let $f_m$ be a function constructed in claim #1 i.e. $f_m$ satisfies (2.2). Without loss of generality we can assume that $\int f_m \eta \, dx = 0$ since (2.2) does not depend on constants. Then

$$f_m = f_{m+1} \text{ on } V_m \text{ for all } m \in \mathbb{N}.$$  

**Proof of Claim #3.** Set $h := f_{m+1} - f_m$ with it $h$ is an element of $L^1(V_m; \mathbb{R})$ and $\int h \text{div} g \, dx = 0 \forall g \in C^1_c(V_m; \mathbb{R})$ since $C^1_c(V_m; \mathbb{R}) \subset C^1_c(V_{m+1}; \mathbb{R})$. The particular choice of $g = \varphi e_i$ for $\varphi \in C^\infty_c(V_m; \mathbb{R})$ leads to $\int h \frac{\partial \varphi}{\partial x_i} \, dx = 0 \forall \varphi \in C^\infty_c(V_m; \mathbb{R})$ and $i = 1, \ldots, n$. This implies $h \in W^{1,1}_{\text{loc}}(V_m; \mathbb{R})$ with $\frac{\partial h}{\partial x_i} = 0$ and therefore constant almost everywhere in $V_m$ i.e. $h(x) \equiv c$. By assumption $c = \int h \eta \, dx = 0$ and therefore $h \equiv 0$ on $V_m$.

In summary we can define $f \in BV_{\text{loc}}(U; \mathbb{R})$ with the desired properties as

$$f(x) = f_m(x) \text{ for any } m \text{ large enough such that } x \in V_m.$$ 

\qed
Chapter 3

Convex functions

We will need some basic estimates for convex functions and they will be presented in this chapter. Again, these results are general knowledge but we will give original proofs.

The following Theorem is a version of Theorem 1 in section 6.3 about convex functions in [EG92].

**Theorem 20.** Let $U \subset \mathbb{R}^n$ be open and convex, $f : U \to \mathbb{R}$ convex then $f$ is locally Lipschitz and there are constants $C$ depending only on $n$, such that

$$|f(x)| \leq \frac{C}{w_n r^n} \int_{B(x,r)} |f| \, dy$$

(3.1)

and if $f$ is $C^1$

$$|Df(x)| \leq \frac{C}{w_n r^{n+1}} \int_{B(x,r)} |f| \, dy$$

(3.2)

for each $x \in U$ such that $B(x, r) \subset U$.

**Remark 5** (Equivalence between the stated version and the result in [EG92]). The stated version looks weaker at first sight since $|f(x)| \leq \sup_{B(x, \frac{r}{2})} |f|$. But in fact they are equivalent because $B(z, \frac{r}{2}) \subset B(x, r)$ $\forall z \in B(x, \frac{r}{2})$ and therefore (3.1) leads to

$$|f(z)| \leq \frac{2^n C}{w_n r^n} \int_{B(z, \frac{r}{2})} |f| \, dy \leq \frac{2^n C}{w_n r^n} \int_{B(x, r)} |f| \, dy \quad \forall z \in B(x, \frac{r}{2}).$$

The same arguments applied to the second estimate gives the original result for $f \in C^1$.

Applying the smoothing argument used at the end of the proof in [EG92] one recovers their version.

**Remark 6** (Estimate for convex functions on compact sets). Let $f : U \to \mathbb{R}$ be a convex functions on an open convex set $U \subset \mathbb{R}^n$. For any compact set $K \subset U$ with $K_{\delta} := \{x \in \mathbb{R}^n : \text{dist}(x, K) < \delta\} \subset U$ for some $\delta > 0$ then there is a constant $C$
depending just on the dimension \( n \) such that
\[
\sup_{x \in K} (|f| + \delta |Df|) \leq \frac{C}{\delta^n} \int_{K_\delta} |f| \, dy. \tag{3.3}
\]
This follows immediately from theorem 20 and \( B(x, \delta) \subset K_\delta \subset U \) for all \( x \in K \) by assumption. The estimate holds for \( f \) not necessarily \( C^1 \) as well. To see this use a smoothing argument (compare with the proof to section 6.3 Theorem 1 in [EG92]): \( f_\epsilon := \rho_\epsilon \ast f \) is still convex for the standard mollifier \( \rho_\epsilon \), apply the estimate proven for the smooth case and take the limit \( \epsilon \to 0 \).

**Lemma 21.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be convex and \( C^2 \), \( U \subset \mathbb{R}^n \) bounded with Lipschitz boundary, then for any \( \delta > 0 \) and any unit vector \( v \in \mathbb{R}^n \)
\[
\int_{U} v \cdot D^2 f v \, dx \leq \frac{1}{\delta} \sup_{x \in \partial U, |y-x| \leq \delta} |f(y) - f(x)| \, \mathcal{H}^{n-1}(\partial U). \tag{3.4}
\]

**Proof.** \( f \) convex implies that its Hessian \( D^2 f \) is a positive semidefinite matrix and therefore \( v \cdot D^2 f(x) v \leq \text{trace}(D^2(f)) = \text{div}(\nabla f(x)) \) for any unit vector \( v \in \mathbb{R}^n \) and \( x \in U \). Furthermore \( f \) convex and \( C^1 \) implies that \( f(y) \geq f(x) + \nabla f(x) \cdot (y - x) \). These inequalities and the divergence theorem with \( \nu \) denoting the unite outer normal of \( \partial U \) lead to
\[
\int_{U} v \cdot D^2 f(x) v \, dx \leq \int_{U} \text{div}(\nabla f(x)) \, dx = \int_{\partial U} \nabla f(x) \cdot \nu \, d\mathcal{H}^{n-1}(x)
\]
\[
= \frac{1}{\delta} \left( \int_{U} \nabla f(x) \cdot \delta \nu \, d\mathcal{H}^{n-1}(x) \right) \leq \frac{1}{\delta} \int_{\partial U} f(x + \delta \nu) - f(x) \, d\mathcal{H}^{n-1}(x)
\]
\[
\leq \frac{1}{\delta} \sup_{x \in \partial U, |y-x| \leq \delta} |f(y) - f(x)| \, \mathcal{H}^{n-1}(\partial U).
\]

\( \square \)

**Corollary 22.** Lemma 21 holds for every \( f : \mathbb{R}^n \to \mathbb{R} \) convex with \( D^2 f \) replaced by its distributional derivative as Radon measure \((\mu_{i,j})\), compare Theorem 7, i.e.
\[
\left( \sum_{i,j=1}^{n} v_i v_j \mu_{i,j} \right)(U) \leq \frac{1}{\delta} \sup_{x \in \partial U, |y-x| \leq \delta} |f(y) - f(x)| \, \mathcal{H}^{n-1}(\partial U) \quad \forall v \in \mathbb{R}^2 \text{ with } |v| = 1 \text{ and } \delta > 0. \tag{3.5}
\]

**Remark 7.** Since \( \mu_{i,i} = e_i \cdot (\mu_{i,i}) e_i \) and for \( i \neq j \)
\[
\mu_{i,j} = \frac{1}{\sqrt{2}} (e_i + e_j) \cdot (\mu_{i,i}) \frac{1}{\sqrt{2}} (e_i + e_j) - \frac{1}{2} \mu_{i,i} - \frac{1}{2} \mu_{j,j},
\]
the obtained bound (3.5) implies
\[
|\mu_{i,j}|(U) \leq \frac{1}{\delta} \sup_{x \in \partial U, |y-x| \leq \delta} |f(y) - f(x)| \, \mathcal{H}^{n-1}(\partial U) \text{ for } i,j \in \{1, \ldots, n\}. \]
Proof. Let $\varphi \in C_c(\mathbb{R}^n; \mathbb{R})$ with $\text{supp}(\varphi) \subseteq U$ and $0 \leq \varphi \leq 1$. Since $\text{supp}(\varphi) \subseteq U$ compact there exists $\epsilon_0 > 0$ such that $\text{dist}(x, \partial U) > 3\epsilon_0$ for all $x \in \text{supp}(\varphi)$. Thus $\varphi_\epsilon := \rho_\epsilon * \varphi \in C^\infty_c(\mathbb{R}^2; \mathbb{R})$ with $\text{supp}(\varphi_\epsilon) \subseteq U$ and $0 \leq \varphi_\epsilon \leq 1$ for all $0 < \epsilon < \epsilon_0$. Furthermore it is straightforward to see that $f_\epsilon := \rho_\epsilon * f$ is still convex and $f_\epsilon \to f$ uniformly. The defining property of the Radon measures \cite{L8} from the first to the second line and the previous lemma gives

$$\sum_{i,j=1}^n \int \varphi v_i v_j d\mu_{i,j} = \sum_{i,j=1}^n \lim_{\epsilon \to 0} \int \varphi_\epsilon v_i v_j d\mu_{i,j}$$

$$= \lim_{\epsilon \to 0} \int \rho_\epsilon * (v \cdot D^2(\rho_\epsilon * \varphi)v) f \ dx = \lim_{\epsilon \to 0} \int \rho_\epsilon * \varphi (v \cdot D^2 f_\epsilon v) \ dx$$

$$\leq \lim_{\epsilon \to 0} \int_U (v \cdot D^2 f_\epsilon v) \ dx \leq \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sup_{x \in \partial U} \frac{|f_\epsilon(y) - f_\epsilon(x)|}{\epsilon} \mathcal{H}^{n-1}(\partial U)$$

$$\leq \frac{1}{\delta} \sup_{x \in \partial U} |f(y) - f(x)| \mathcal{H}^{n-1}(\partial U).$$

\[ \square \]

**Lemma 23.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex, $A \subseteq \mathbb{R}^n$ with vanishing $n - 1$ dimensional Hausdorff measure i.e. $\mathcal{H}^{n-1}(A) = 0$ then for any unit vector $v \in \mathbb{R}^n$

$$(v \cdot (\mu_{i,j})v)(A) = 0 \quad (3.6)$$

where we used the abbreviation $(v \cdot (\mu_{i,j})v) = \sum_{i,j=1}^n v_i v_j \mu_{i,j}$.

**Remark 8.** With the same argument as in the last remark \cite{3.6} implies then that

$$|\mu_{i,j}|(A) = 0 \text{ for } i,j \in \{1, \ldots, n\} \text{ and } A \subseteq \mathbb{R}^n \text{ with } \mathcal{H}^{n-1}(A) = 0.$$  

**Proof.** By countable additivity we can assume that $A$ is contained in a compact set $K \subseteq \mathbb{R}^n$. Fix $\delta > 0$ and fix the compact set $K^\delta := \{x \in \mathbb{R}^2 : \text{dist}(x, K) \leq \delta\}$.

$$M := \sup_{x \in K^\delta} |f(y) - f(x)|$$

is finite since $f$ is continuous. By definition of the Hausdorff measure $\mathcal{H}^{n-1}$ we can choose sets $C_1, C_2, \ldots$ such that $A \subseteq \bigcup_{k=1}^\infty C_k$, $C_k \cap A \neq \emptyset$ for all $k$, $r_k = \text{diam}(C_k) \leq \delta$ and

$$\sum_k w_{n-1} \left(\frac{r_k}{2}\right)^{n-1} < \frac{\delta}{2^{n-1} n w_n M} \epsilon.$$  

Now take $x_k \in C_k \cap A$, so that $A \subseteq \bigcup_{k=1}^\infty B(x_k, r_k)$ and we conclude from \cite{3.5} consid-
ering that $\overline{B(x_k, r_k)} \subset K^3$ for all $k$

\[
(v \cdot (\mu_{i,j})v) (A) \leq \sum_k (v \cdot (\mu_{i,j})v) (B(x_k, r_k))
\leq \sum_k \frac{1}{\delta} \sup_{x \in \partial B(x_k, r_k)} |f(y) - f(x)| \mathcal{H}^{n-1}(\partial B(x_k, r_k))
\leq \sum_k \frac{M}{\delta} n w_n r_k^{n-1} = \frac{2^{n-1} n w_n M}{w_{n-1} \delta} \sum_k w_{n-1} \left( \frac{r_k}{2} \right)^{n-1}
< \epsilon.
\]

Since $\epsilon > 0$ has been arbitrary we have proved the lemma. $\square$
Chapter 4

Proof of the representation theorem

With the theory established so far we are now able to prove our main theorem. We are going to present two different proofs for the part where the second derivative of a convex function is a stationary varifold. The first proof works for all dimensions and is based on Schwarz’s theorem about the commutation of partial derivatives; the other works only in the plane since the key calculations cannot be directly generalised to higher dimensions. The second one uses approximation by maxima over finitely many hyperplanes and is presented in the subsequent section. It provides some insight into how varifolds represented by second derivatives of convex functions look.

1. Proof of Theorem 6. Let $U$ be an open convex subset of $\mathbb{R}^{n+1}$, $f : U \to \mathbb{R}$ a convex function and $(\mu_{i,j})$ the Radon measure determined by its second derivative, compare Theorem 7. For any smooth vector field $\psi : U \to \mathbb{R}^{n+1}$ with compact support we encounter using the defining equation (1.8) and $\frac{\partial \psi_i}{\partial x_j} \in C^2_c(U; \mathbb{R})$

$$
\sum_{i,j=1}^{n+1} \left( \int_U \frac{\partial \psi_i}{\partial x_i} d\mu_{j,j} - \int_U \frac{\partial \psi_i}{\partial x_j} d\mu_{i,j} \right) = \sum_{i,j=1}^{n+1} \left( \int_U f \frac{\partial^3 \psi_i}{\partial x_j \partial x_i} dx - \int_U f \frac{\partial^3 \psi_i}{\partial x_i \partial^2 x_j} dx \right) = 0.
$$

For an arbitrary $C^1$-vector field $\phi : U \to \mathbb{R}^{n+1}$ compactly supported take a sequence $\{\psi^{(k)}\}_{k \in \mathbb{N}}$ of smooth compactly supported vector fields $\psi^{(k)} : U \to \mathbb{R}^{n+1}$ with $\psi^{(k)} \to \phi$ as $k \to \infty$ i.e. $\lim_{k \to \infty} \| \frac{\partial \psi_i^{(k)}}{\partial x_j} - \frac{\partial \psi_i}{\partial x_j} \|_\infty = 0$ $(i,j = 1, \ldots, n+1)$. This implies

$$
\sum_{i,j=1}^{n+1} \left( \int_U \frac{\partial \phi_i}{\partial x_i} d\mu_{j,j} - \int_U \frac{\partial \phi_i}{\partial x_j} d\mu_{i,j} \right) = \lim_{k \to \infty} \sum_{i,j=1}^{n+1} \left( \int_U \frac{\partial \psi_i^{(k)}}{\partial x_i} d\mu_{j,j} - \int_U \frac{\partial \psi_i^{(k)}}{\partial x_j} d\mu_{i,j} \right) = 0
$$

proving that the stationary condition (1.3) holds and therefore the theorem.

The proof for the converse direction for $n = 1$ uses the same techniques as the
proof to Lemma \[19\] So let \((\mu_{i,j})\) with \(i,j = 1,2\) be the second moment of a stationary 1-varifold and therefore satisfying \([1.3]\). We have to show that there is a convex function \(f : U \to \mathbb{R}\) such that

\[
\int_U f \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \, dx = \int_U \varphi \, d\mu_{i,j} \quad \forall \varphi \in C^2_c(U; \mathbb{R}).
\] (4.1)

Choose a sequence of non empty open subsets \(W_m \subset U\) with the following properties

1. \(W_m\) is a bounded open convex subset for every \(m\);
2. \(W_m \subset W_{m+1}\) for every \(m\) and \(\bigcup_{m=1}^{\infty} W_m = U\);
3. for every \(m\) there is \(\delta_m > 0\) such that \((W_m)_{3\delta_m} = \{x \in \mathbb{R}^n : \text{dist}(x,W_m) < 3\delta_m\} \subset U\) and \(\delta_{m+1} \leq \delta_m\).

Note that this implies that \(\overline{W}_m\) and \((\overline{W}_m)_\delta = \{x \in \mathbb{R}^n : \text{dist}(x,W_m) \leq \delta\}\) for \(0 \leq \delta \leq 2\delta_m\) are compact subset of \(U\). We denote the sets \((W_m)_{\delta_m} = \{x \in \mathbb{R}^n : \text{dist}(x,W_m) < \delta_m\}\) for \(0 \leq \delta \leq 2\delta_m\).

An admissible sequence would be \(W_m := \{x \in U : \text{dist}(x,\partial U) > \frac{\delta_m}{m}\} \cap B(0, m)\) together with \(\delta_m = \frac{1}{m}\).

Fix \(\eta \in C^\infty_c(V_1; \mathbb{R})\) nonnegative with \(\int \eta \, dx = 1\).

We are considering every \(m\) separately and will put the results obtained for every single \(m\) together at the end.

Consider in the following a fixed \(m\).

As before the particular choice of \(V_m\) enables us to use lemma \[15\] to approximate the Radon measures on \(V_m\); hence for every \(i,j = 1,2\) there is a sequence \(\{F^\varepsilon_{i,j}\}_{0 < \varepsilon < \delta_m} \subset C^\infty(V_m; \mathbb{R})\) such that for all \(\varphi \in C^\infty_c(V_m; \mathbb{R})\) and \(i,j = 1,2\)

\[
\int_{V_m} \varphi \, F^\varepsilon_{i,j} \, dx = \int_{V_m} \rho_\varepsilon * \rho_\varepsilon * \varphi \, d\mu_{i,j}.
\]

Claim \#1. The matrix \((F^\varepsilon_{i,j})\) is symmetric, positive semidefinite for every \(0 < \varepsilon < \delta_m\).

Proof of Claim \#1. Symmetry holds since \(\mu_{1,2} = \mu_{2,1}\), see lemma \[5\]. Choosing in the same lemma \(\psi = \sqrt{\varphi} a\) for any positive \(\varphi \in C_c(U; \mathbb{R})\) and \(a = (a_1, a_2) \in \mathbb{R}^2\) we encounter

\[
0 \leq \sum_{i,j=1}^{2} a_i a_j \int_U \varphi \, d\mu_{i,j}
\]

Take \(\varphi = \rho_\varepsilon * \rho_\varepsilon * \phi\) that is non-negative for \(\phi \in C_c(V_m; \mathbb{R})\) with \(\phi \geq 0\). The positive
semidefiniteness is proven since this leads to

$$0 \leq \int_U \phi \left( \sum_{i,j=1}^2 a_i a_j F_{i,j}^\epsilon \right) dx.$$

Claim #2. For the fixed $m$ there are functions $F_1^\epsilon, F_2^\epsilon \in C^\infty(V_m; \mathbb{R})$ such that for $i, j = 1, 2$

$$\frac{\partial F_i^\epsilon}{\partial x_j} = F_{j,i}^\epsilon \quad \text{and} \quad (4.2)$$

$$\int_U F_i^\epsilon \eta \, dx = 0. \quad (4.3)$$

Proof of Claim #2. It is enough to prove

$$\frac{\partial F_1^\epsilon,i}{\partial x_2} - \frac{\partial F_2^\epsilon,i}{\partial x_1} = 0 \quad \text{for all } 0 < \epsilon < \delta_m \text{ and } i = 1, 2 \quad (4.4)$$

since $V_m$ is a star-shaped open subset of $\mathbb{R}^n$, thus every closed covector field is exact. We can assume without loss of generality that (4.3) is satisfied as well by choosing an appropriate constant of integration.

The stationary condition (1.3) for $n = 1$ is that for any \( \psi \in C^1_c(U; \mathbb{R}^2) \)

$$\left( \int_U \frac{\partial \psi}{\partial x_2} \, d\mu_{1,1} - \int_U \frac{\partial \psi}{\partial x_1} \, d\mu_{2,1} \right) + \left( \int_U \frac{\partial \psi}{\partial x_2} \, d\mu_{2,2} - \int_U \frac{\partial \psi}{\partial x_1} \, d\mu_{1,2} \right) = 0.$$

In particular for \( \psi = \rho_\epsilon \ast \rho_\epsilon \ast \varphi \) with \( \varphi = (\varphi_1, \varphi_2) \in C^\infty_c(V_m; \mathbb{R}^2) \) we obtain according to (2.4)

$$0 = \left( \int_U \rho_\epsilon \ast \rho_\epsilon \ast \frac{\partial \varphi_2}{\partial x_2} \, d\mu_{1,1} - \int_U \rho_\epsilon \ast \rho_\epsilon \ast \frac{\partial \varphi_2}{\partial x_1} \, d\mu_{2,1} \right)$$

$$+ \left( \int_U \rho_\epsilon \ast \rho_\epsilon \ast \frac{\partial \varphi_1}{\partial x_2} \, d\mu_{2,2} - \int_U \rho_\epsilon \ast \rho_\epsilon \ast \frac{\partial \varphi_1}{\partial x_1} \, d\mu_{1,2} \right)$$

$$= \left( \int_{V_m} \frac{\partial \varphi_2}{\partial x_2} F_{1,1}^\epsilon - \frac{\partial \varphi_2}{\partial x_1} F_{2,1}^\epsilon \, dx \right) + \left( \int_{V_m} \frac{\partial \varphi_1}{\partial x_2} F_{2,2}^\epsilon - \frac{\partial \varphi_1}{\partial x_1} F_{1,2}^\epsilon \, dx \right).$$

All involved functions are smooth and \( \varphi \) is compactly supported so we can integrate by parts.

$$0 = \int_{V_m} \varphi_2 \left( \frac{\partial F_{1,1}^\epsilon}{\partial x_2} - \frac{\partial F_{2,1}^\epsilon}{\partial x_1} \right) \, dx + \int_{V_m} \varphi_1 \left( \frac{\partial F_{1,2}^\epsilon}{\partial x_2} - \frac{\partial F_{2,2}^\epsilon}{\partial x_1} \right) \, dx$$

for any \( \varphi = (\varphi_1, \varphi_2) \in C^\infty_c(V_m; \mathbb{R}^2) \) proving claim #2.
As a consequence of (4.2) and the symmetry of the matrix $(F_{i,j})$ we encounter

\[
\frac{\partial F^\epsilon_1}{\partial x_2} - \frac{\partial F^\epsilon_2}{\partial x_1} = F^\epsilon_{2,1} - F^\epsilon_{1,2} = 0.
\]

Thus $(F^\epsilon_1, F^\epsilon_2) : V_m \to \mathbb{R}^2$ is itself an exact covector field. So there is $f_\epsilon \in C^\infty(V_m; \mathbb{R})$ satisfying

\[
\frac{\partial f_\epsilon}{\partial x_i} = F^\epsilon_i \quad \text{for } i = 1, 2 \quad \text{and} \quad \int_{V_m} f_\epsilon \eta \, dx = 0. \tag{4.5}
\]

$f_\epsilon : V_m \to \mathbb{R}$ is a convex function for every $0 < \epsilon < \delta_m$ since its Hessian $\frac{\partial^2 f_\epsilon}{\partial x_i \partial x_j} = F^\epsilon_{i,j}$ is positive semidefinite.

**Claim #3.** For the chosen $m$ there is $f_m : W_m \to \mathbb{R}$ convex satisfying (4.1) on $W_m$ i.e.

\[
\int_{W_m} f_m \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \, dx = \int_{W_m} \varphi \, d\mu_{i,j} \quad \forall \varphi \in C^2_c(W_m; \mathbb{R}),
\]

where the second derivative of $f_m$ coincides with the the second moments $(\mu_{i,j})$ of the stationary 1-varifold.

**Proof of Claim #3.** Firstly we will establish the following a priori estimate. There is a constant $C > 0$ depending only on $m$ and $n$ such that for $0 < \epsilon < \delta_m^2$

\[
\sup_{x \in W_m} (|f_\epsilon| + \delta_m |Df_\epsilon|) \leq \frac{C}{(\delta_m)^n} (\mu_{1,1} + \mu_{2,2}) ((V_m)\delta_m) < \infty. \tag{4.6}
\]

As a consequence of the properties of $V_m$, lemma [14] and lemma [18] Poincaré’s inequality holds on \( \{ u \in W^{1,1}(V_m; \mathbb{R}) : \int u \eta \, dx = 0 \} \) i.e.

\[
\int_{V_m} |u| \, dx \leq C \int_{V_m} |Du| \, dx \quad \forall u \in W^{1,1}(V_m; \mathbb{R}) \text{ with } \int u \eta \, dx = 0.
\]

$f_\epsilon, F^\epsilon_1, F^\epsilon_2$ are functions in this set for $0 < \epsilon < \delta_m^2$ so that due to (4.2) and (4.5)

\[
\int_{V_m} |f_\epsilon| \, dx \leq C \sum_{i=1}^{2} \int_{V_m} |F^\epsilon_i| \, dx
\]

\[
\int_{V_m} |F^\epsilon_i| \, dx \leq C \sum_{j=1}^{2} \int_{V_m} |F^\epsilon_{j,i}| \, dx \quad (i = 1, 2).
\]

For a symmetric positive semidefinite matrix $S = (s_{i,j}) \in \mathbb{R}^{n \times n}$ one has $|s_{i,j}| \leq \text{trace } S$ for all $i, j$. $(F^\epsilon_{i,j})$ is such a symmetric positive semidefinite matrix. Furthermore as established in Lemma [15] for $0 < \epsilon < \delta_m^2$ and $i = 1, 2$

\[
\int_{V_m} |F^\epsilon_{i,i}| \leq \mu_{i,i}((V_m)\delta_m) \leq \mu_{i,i}((V_m)\delta_m).
\]
In summary we have shown that for a constant $C$ just depending on $n, m$

$$\int_{V_m} |f_i| \, dx \leq 3C^2 \sum_{i=1}^{2} \int_{V_m} |F_{i,i}^m| \, dx \leq 3C^2 (\mu_{1,1} + \mu_{2,2}) ((V_m)_{\delta_m}).$$

The estimate for convex functions on compact sets in remark 6 applied to the convex set $W_m$ together with $(W_m)_{\delta_m} = V_m$ shows (4.6).

We can apply the Arzelá-Ascoli theorem to the sequence $\{f_\epsilon\}_{0 < \epsilon < \frac{\delta_m}{2}}$ as an uniformly bounded sequence of equicontinuous functions on $W_m$ following consequently from (4.6).

Let $f_{\epsilon_k}$ be an extracted subsequence with $f_{\epsilon_k} \to f_m$ on $W_m$ uniformly as $k \to \infty$. The functions $f_{\epsilon_k}$ are convex so is $f_m$. Furthermore for any $\varphi \in C^2_c(W_m; \mathbb{R})$ and $\nu_{i,j}^{(m)}$ denoting the second derivative of the obtained convex function $f_m$ in the sense of theorem 7

$$\int_{W_m} \varphi \, d\nu_{i,j}^{(m)} = \int_{W_m} f_m \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \, dx = \lim_{k \to \infty} \int_{W_m} f_{\epsilon_k} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \, dx = \lim_{k \to \infty} \int_{W_m} \varphi F_{i,j}^{\epsilon_k} \, dx = \lim_{k \to \infty} \int_{W_m} \rho_{\epsilon_k} * \rho_{\epsilon_k} * \varphi \, d\mu_{i,j}$$

$$= \int_{W_m} \varphi \, d\mu_{i,j}.$$

This proves claim #3.

$m$ has been fixed but arbitrary so that for every $m \in \mathbb{N}$ there is a convex function established in claim #3 $f_m : W_m \to \mathbb{R}$ satisfying (4.1) on $W_m$. This condition is invariant under change by affine function so that we can assume without loss of generality that for every $m \in \mathbb{N}$ $\int \frac{\partial f_m}{\partial x_i} \eta \, dx = 0$ (i = 1, 2) and $\int f_m \eta \, dx = 0$. Recall that $f_m$ convex implies that $f_m \in W^{1,1}(W_m; \mathbb{R})$.

Claim #4.

$$f_m = f_{m+1} \text{ on } W_m \text{ for all } m \in \mathbb{N}.$$  

Proof of Claim #4. Set $h := f_{m+1} - f_m$. $h$ is therefore an element of $W^{1,1}(W_m; \mathbb{R})$ and for all $\varphi \in C^2_c(W_m; \mathbb{R})$

$$\int h \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \, dx = 0 \quad (i, j = 1, 2)$$

since $C^2_c(W_m; \mathbb{R}) \subset C^2_c(W_{m+1}; \mathbb{R})$. Thus $h$ is actually an element of $W^{2,1}(W_m; \mathbb{R})$ with $\frac{\partial^2 h}{\partial x_i \partial x_j} = 0$ for $i, j = 1, 2$. Hence $h$ is affine almost everywhere in $W_m$ i.e. $h(x) = ax + b$. By assumption $a_i = \int \frac{\partial h}{\partial x_i} \eta \, dx = 0$ (i = 1, 2). Consequently $a \equiv 0, h(x) = b$ and $b = \int h \eta \, dx = 0$ therefore $h \equiv 0$ on $W_m$.  

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In summary we can define the convex function \( f : U \to \mathbb{R} \) with the desired properties as
\[
    f(x) = f_m(x) \text{ for any } m \text{ large enough such that } x \in W_m.
\]

### 4.1 More geometric investigation of the representation theorem

As outlined in the introduction, we present the second proof of the part of theorem \( \square \) that states that the second derivatives of convex functions are second moments of stationary varifolds. This one is more geometric and gives insight into how the second derivatives of convex functions represent varifolds. The drawback of it is that it only works in the plane.

The rough idea is that one can calculate explicitly the 1-varifold represented by the convex function that is a maximum over finitely many hyperplanes. Approximation of an arbitrary convex function by such simpler ones will give us the general case.

#### 4.1.1 Maxima over finitely many hyperplanes

In this section we are going to calculate the 1-varifold represented by finitely many hyperplanes.

Let \( \{g_i\}_{i=1}^N \) be a family of hyperplanes i.e \( g_i(x) = a_i \cdot x + b_i \) with \( a_i \in \mathbb{R}^2 \) and \( b_i \in \mathbb{R} \). Without loss of generality we can assume that \( a_i \neq a_j \) for \( i \neq j \). The convex function of interest is
\[
    f(x) := \sup_{i=1,\ldots,N} g_i(x).
\]

\( f \) is indeed convex as a supremum over convex functions. \( \mu_{i,j} \) denotes the via \( (1.8) \) determined Radon measure, the second derivative of \( f \). In the following we will calculate \( (\mu_{i,j}) \) explicitly.

Let \( G_i := \{(x, g_i(x)) : x \in \mathbb{R}^2\} \) the hyperplane in \( \mathbb{R}^3 \) generated by \( g_i \) and \( F := \{(x, f(x)) : x \in \mathbb{R}^2\} \) denotes the graph of \( f \). All these sets are closed as graphs of continuous functions. We note that \( G_i \neq G_j \) if \( i \neq j \) by construction. Therefore \( G_i \cap G_j \) for \( i \neq j \) is the line in \( \mathbb{R}^3 \) where \( g_i \) and \( g_j \) coincide. Thus the union of these lines
\[
    \bigcup_{1 \leq i < j \leq N} (G_i \cap G_j)
\]
has Hausdorff dimension 1 so has therefore the following closed set

\[ \tilde{\Sigma} := F \cap \bigcup_{1 \leq i < j \leq N} (G_i \cap G_j). \]

Furthermore sets of the form \( G_i \cap G_j \cap G_k \) for \( i < j < k \) contain at most one point so that

\[ \tilde{S} := F \cap \bigcup_{1 \leq i < j < k \leq N} (G_i \cap G_j \cap G_k) \]

is finite.

Let \( \Pi : \mathbb{R}^3 \to \mathbb{R}^2 \) be the orthogonal projection onto the first two components. We define the sets

\[ \Sigma := \Pi(\tilde{\Sigma}) \quad \quad S := \Pi(\tilde{S}). \] (4.8)

Since \( H^s(\Pi(A)) \leq H^s(A) \) for all \( s \geq 0 \), e.g. section 2.4.1 Corollary 1 in [EG92], \( \Sigma \) has Hausdorff dimension 1. In fact it is easy to see that \( \Sigma \) is a subset of the set of lines

\[ \bigcup_{1 \leq i < j \leq N} \{ x \in \mathbb{R}^2 : g_i(x) = g_j(x) \}. \]

One observes that the set \( \mathbb{R}^2 - \Sigma \) can be characterised as the set of points where the supremum in (4.7) is taken by a single hyperplane

\[ \mathbb{R}^2 - \Sigma = \{ x \in \mathbb{R}^2 : \exists \text{ unique } i = i(x) \in \{1, \ldots, N\} \text{ s.t. } f(x) = g_i(x) > g_j(x) \forall j \neq i \}. \]

It is open because for \( x \in \mathbb{R}^2 - \Sigma \) and \( i = i(x) \) one has \( g_i(x) > g_j(x) \) for every \( j \neq i \). Thus there is an open neighbourhood \( V_j \) of \( x \) such that \( g_i(y) > g_j(y) \forall y \in V_j. \) \( V = \bigcap_{j=1}^N V_j \) is open and \( g_i(y) > g_j(y) \forall j \neq i \) and \( y \in V \). Therefore \( V \subset U \) showing that \( U \) is open. This is summarised in the first part of the following lemma.

**Lemma 24.** Let \( f = \sup_{i=1, \ldots, N} g_i \) as above be the supremum over finitely many hyperplanes i.e. \( g_i(x) = a_i \cdot x + b_i \) with \( a_i \in \mathbb{R}^2, b_i \in \mathbb{R} \) and \( a_i \neq a_j \) for \( i \neq j \). Then the above defined subsets \( \mathbb{R}^2 - \Sigma, \Sigma, S \) of \( \mathbb{R}^2 \) have the following properties

1. \( \mathbb{R}^2 - \Sigma \) is open and has full Lebesgue measure \( \mathcal{L}^2 \);

2. \( \Sigma \) is a closed set with Hausdorff dimension 1 contained in the collection of lines \( \bigcup_{1 \leq i < j \leq N} \{ x \in \mathbb{R}^2 : g_i(x) = g_j(x) \} \);

3. \( S \subset \Sigma \) is finite.

Furthermore every \( x \in \Sigma - S \) has an open neighbourhood \( U \) such that

\[ f(x) = \max\{g_i(x), g_j(x)\} \text{ on } U \text{ for precisely two } i \neq j \in \{1, \ldots, N\}. \] (4.9)

Thus the situation sketched in figure 4.1 holds. Every \( x \in S \) has a open neighbourhood such that we are in the situation of figure 4.2.
Proof. The last statement remains to be proven. Let $x \in \Sigma - S$ then there are precisely two elements $i \neq j \in \{1, \ldots, N\}$ such that $f(x) = g_i(x) = g_j(x)$ otherwise it would be an element of $S$. $h := \max\{g_i, g_j\}$ and $k := \sup_{l \in \{1, \ldots, N\} \setminus \{i,j\}} g_l$ are continuous functions and by construction $f(x) = h(x) > k(x)$. Thus there is an open neighbourhood $U$ of $x$ such that $h(y) > k(y)$ for all $y \in U$. We established (4.9) since $f = \max\{h, k\}$. The last part is trivial since $S$ is finite. \qed

Lemma 25. In case of the current situation, $f = \sup_{i=1, \ldots, N} g_i$, there is a $\mathcal{H}^1$-measurable $\mathbb{R}^{2 \times 2}$-valued function $(\sigma_{i,j}) : \Sigma \to \mathbb{R}^{2 \times 2}$ such that the second derivative of $f$ in terms of the signed Radon measure $(\mu_{i,j})$ is given as

$$
\mu_{i,j} = \sigma_{i,j} d\mathcal{H}^1 \setminus \Sigma
$$

whereas

$$
(\sigma_{i,j}) : \Sigma \to \mathbb{R}^{2 \times 2}
$$

$$
(\sigma_{i,j})(x) = |a_k - a_l| n \otimes n
$$

for $x \in U \cap \Sigma, U \subset \mathbb{R}^2$ open and $f = \max\{g_k, g_l\}$ on $U$, $n := \frac{a_k - a_l}{|a_k - a_l|}$. 

Remark 9. As a consequence of the previous lemma, $\sigma_{i,j}$ is well-defined on $\Sigma - S$. This is sufficient since $\mathcal{H}^1(S) = 0$. We will adopt the convention that $(\sigma_{i,j}) \equiv 0$ on $\mathbb{R}^2 - \Sigma$, compare \cite{Sim83} chapter 4.
Proof. Claim #1. The support of the measures $\mu_{i,j}$ is contained in $\Sigma$ and therefore in a set of Hausdorff dimension 1.

Proof of Claim #1. Let $\varphi \in C^2_c(\mathbb{R}^2; \mathbb{R})$ with $\text{supp}(\varphi) \subset \mathbb{R}^2 - \Sigma$. Let us assume that the support of $\varphi$ is contained in an open set $U$ such that $f = g_i$ on $U$ for some unique $i$. $g_i$ is an affine function and therefore

$$
\int_{\mathbb{R}^2} \varphi \, d\mu_{k,l} = \int_U f \frac{\partial^2 \varphi}{\partial x_k \partial x_l} \, dx = \int_U g_i \frac{\partial^2 \varphi}{\partial x_k \partial x_l} \, dx = \int_U \frac{\partial^2 g_i}{\partial x_k \partial x_l} \varphi \, dx = 0.
$$

In case of a general $\varphi$ with $\text{supp}(\varphi) \subset U$ consider that every $x \in \mathbb{R}^2 - \Sigma$ has neighbourhood $U_x$ such that $f = g_i$ on $U_x$ for a unique $i \in \{1, \ldots, N\}$. $\text{supp}(\varphi)$ is compact and so it has a finite open cover $\text{supp}(\varphi) \subset \bigcup_{m=1}^M U_{x_m}$. Let $\{\eta_m\}_{m=1}^M$ be an associated smooth partition of unity. So $\varphi = \sum_{m=1}^M \eta_m \varphi$ with $\eta_m \varphi \in C^2_c(\mathbb{R}^2; \mathbb{R})$ and $\text{supp}(\eta_m \varphi) \subset U_{x_m}$. Thus

$$
\int_{\mathbb{R}^2} \varphi \, d\mu_{k,l} = \sum_{m=1}^M \int_{\mathbb{R}^2} \eta_m \varphi \, d\mu_{k,l} = 0
$$

by the previous calculation.

Claim #2. Let $x \in \Sigma - S$ and $U$ be an open neighbourhood of $x$ such that $f =
max\{g_i, g_j\} on U, compare figure 4.1. Then
\[
(\mu_{k,l})_{i,j} U = |a_i - a_j| n \otimes n \mathcal{H}^1(U \cap \Sigma) \text{ with } n = \frac{a_i - a_j}{|a_i - a_j|}. \tag{4.12}
\]

Proof of Claim #2. Consider that \( U \cap \Sigma = \{ x \in U : g_i(x) = g_j(x) \} \).
Without loss of generality we can assume that \( x = 0 \). To simplify notation we assume \( i, j = 1, 2 \). Thus
\[
g_l(x) = a_l \cdot x \text{ for } l = 1, 2 \text{ and } n = \frac{a_1 - a_2}{|a_1 - a_2|}. \quad g_1(s n) - g_2(s n) = s(a_1 - a_2) \cdot n = s|a_1 - a_2| \text{ for } s \in \mathbb{R} \text{ and therefore } g_1(s n) \geq g_2(s n) \text{ for } s \geq 0. \]
Set \( t = (0_1 0_1) n; \) hence \( \{ t, n \} \) is an orthonormal basis for \( \mathbb{R}^2 \) and the matrix \( O = (t, n) \) is orthogonal.
Let \( \varphi \in C^2_b(\mathbb{R}^2; \mathbb{R}) \) with \( \text{supp}(\varphi) \subset U \). Set \( \psi(z) = \varphi(O z) \) this implies that \( D^2 \varphi(x) = O D^2 \varphi(O x)^t O^t \). Let \( c_l = O^t a_l \) for \( l = 1, 2 \) then \( c_1 - c_2 = |a_1 - a_2| e_2 \). Writing \( v \cdot (\mu_{i,j}) w \) instead of \( \sum_{i,j=1,2} v_i w_j \mu_{i,j} \) for \( v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2 \) one obtains performing a change of variables \( x = O z \)
\[
\int \varphi(x) \, d(n \cdot (\mu_{i,j}) n) = \int_U n \cdot D^2 \varphi(x) n \, f(x) \, dx = \int_U n \cdot (O D^2 \varphi(z) O^t n) \, f(O z) \, dz
= \int_{U \cap \{ z_2 \geq 0 \}} \frac{\partial^2 \varphi}{\partial z_2^2}(z \cdot c_1 \cdot z) \, dz + \int_{U \cap \{ z_2 \leq 0 \}} \frac{\partial^2 \varphi}{\partial z_2^2}(z \cdot c_2 \cdot z) \, dz
\]
Integration by parts taking into account that the boundary terms over \( \partial U \) are zero since \( \text{supp}(\psi) \subset U \) one obtains
\[
\int \varphi(x) \, d(n \cdot (\mu_{i,j}) n)
= -\int_{U \cap \{ z_2 = 0 \}} \frac{\partial \psi}{\partial z_2}(z \cdot c_1 \cdot z) \, d\mathcal{H}^1(z) - \int_{U \cap \{ z_2 \geq 0 \}} \frac{\partial \psi}{\partial z_2}(z \cdot c_1 \cdot e_2) \, dz
+ \int_{U \cap \{ z_2 = 0 \}} \frac{\partial \psi}{\partial z_2}(z \cdot c_2 \cdot z) \, d\mathcal{H}^1(z) - \int_{U \cap \{ z_2 \leq 0 \}} \frac{\partial \psi}{\partial z_2}(z \cdot c_2 \cdot e_2) \, dz
= \int_{U \cap \{ z_2 = 0 \}} \psi(z \cdot c_1 \cdot e_2) \, d\mathcal{H}^1(z) - \int_{U \cap \{ z_2 = 0 \}} \psi(z \cdot c_2 \cdot e_2) \, d\mathcal{H}^1(z)
= \int_{U \cap \{ z_2 = 0 \}} \psi(z \cdot |a_1 - a_2|) \, d\mathcal{H}^1(z)
\]
applying the divergence theorem and considering the fact that \( -(c_1 \cdot z) + (c_2 \cdot z) = -|a_1 - a_2| z_2 = 0 \) on \( \{ z_2 = 0 \} \).
For \( v \in \{ t, n \} \) we can calculate

\[
\int \varphi(x) \, d(t \cdot (\mu_{i,j})v) = \int_U t \cdot D^2 \varphi(x) v f(x) \, dx = \int_U t \cdot (OD^2 \psi(x) O' v) \, f(Oz) \, dz
\]

\[
= \int_{U \cap \{ z_2 \geq 0 \}} \frac{\partial^2 \psi}{\partial z_1 \partial z_l}(z) \, (c_1 \cdot z) \, dz + \int_{U \cap \{ z_2 \leq 0 \}} \frac{\partial^2 \psi}{\partial z_1 \partial z_l}(z) \, (c_2 \cdot z) \, dz
\]

\[
= -\int_{U \cap \{ z_2 \geq 0 \}} \frac{\partial \psi}{\partial z_l}(z) \, (c_1 \cdot e_1) \, dz - \int_{U \cap \{ z_2 \leq 0 \}} \frac{\partial \psi}{\partial z_l}(z) \, (c_2 \cdot e_1) \, dz
\]

\[
= \int_{U \cap \{ z_2 = 0 \}} \psi(z) \, (c_1 \cdot e_1)(e_1 \cdot e_2) \, d\mathcal{H}^1(z) - \int_{U \cap \{ z_2 = 0 \}} \psi(z) \, (c_2 \cdot e_1)(e_1 \cdot e_2) \, d\mathcal{H}^1(z)
\]

\[
= 0
\]

by firstly integrating by parts then applying the divergence theorem and finally \((c_1 \cdot e_1) - (c_2 \cdot e_2) = 0\). The boundary terms appearing are just over \( \partial U \) and therefore zero.

With the second equation we determined \((\mu_{i,j})_U\) completely as \((4.12)\).

The general lemma \([23]\) shows that \((\mu_{i,j})\) does not put any mass on sets with \(\mathcal{H}^1\)-measure zero. Consequently we do not have to consider \(S\) at all. Nonetheless the following straightforward calculation proves the same without using abstract theory.

**Claim \#3.** Let \( x \in S \) and \( r > 0 \) be small enough. Such that \( f \) looks on \( B(x, r) \) as sketched in figure \([4.2]\). Then

\[
(\mu_{i,j})_U B(x, r) = \sum_{l=1}^{m} |a_{i_l} - a_{i_l}| \, n_{i_{l+1}i_l} \otimes n_{i_{l+1}i_l} \mathcal{H}^1(\{g_{i_{l+1}} = g_{i_l}\} \cap E_l)
\]

(4.13)

with \( n_{i_{l+1}i_l} = \frac{a_{i_{l+1}} - a_{i_l}}{|a_{i_{l+1}} - a_{i_l}|} \) and \( i_{m+1} = i_1 \).

**Proof of Claim \#3.** Simplifying notation we may assume that \( x = 0 \) and \( i_l = l \) for \( l = 0, \ldots, m \). Let \( \varphi \in C_c^2(\mathbb{R}^2; \mathbb{R}) \) with \( \text{supp}(\varphi) \subset B(0, r) \) and \( v, w \in \mathbb{R}^2 \) two arbitrary vectors. \( E_l \) is the circle segment containing just \( \{g_{i_{l+1}} = g_{i_l}\} \cap \Sigma \cap B(0, r) \) as illustrated in figure \([4.3]\). With it \( B(0, r) \) is the disjoint union \( B(0, \epsilon) \cup \bigcup_{l=1}^{m} (B(0, r) - B(0, \epsilon)) \cap E_l \).

Thus

\[
\int \varphi \, d(v \cdot (\mu_{i,j})w) = \int_{B(0, \epsilon)} \varphi \, d(v \cdot (\mu_{i,j})w) + \sum_{l=0}^{m} \int_{(B(0, r) - B(0, \epsilon)) \cap E_l} \varphi \, d(v \cdot (\mu_{i,j})w).
\]
Based on claim #2 we obtain using \( n_l = a_l + 1 - a_l \). 

\[
\int_{(B(0,r)-B(0,\epsilon))\cap E_l} \varphi (v \cdot n_l) w dH^1.
\]

Therefore we obtain the claim by taking the limit \( \epsilon \to 0 \):

\[
\int \varphi d(v \cdot (\mu_{i,j}) w)
= \int_{B(0,r)} v \cdot D^2 \varphi w f dx + \sum_{l=1}^{m} \int_{(B(0,r)-B(0,\epsilon))\cap E_l \cap \{g_{l+1} = g_l\}} \varphi (v \cdot n_l) w dH^1.
\]

To close the whole argument let \( \varphi \in C^2_c(\mathbb{R}^2; \mathbb{R}) \) be arbitrary. For every \( x \in \Sigma - S \) there is an open neighbourhood \( U_x \) such that \( f = \max\{g_i, g_j\} \) on \( U \) for some \( i \neq j \in \{1, \ldots, N\} \) i.e. the situation of claim #2 holds on \( U_x \) and for the finitely many \( x \in S \) there are \( r_x > 0 \) such that claim #3 applies to \( B(x, r_x) \), compare lemma [24]. Since \( \text{supp}(\varphi) \) is compact there is a finite open subcover i.e.

\[
\text{supp}(\varphi) \subset (\mathbb{R}^2 - \Sigma) \cup \bigcup_{x \in S} B(x, r_x) \cup \bigcup_{m=1}^{M} U_{x_m}.
\]

Let \( \{\eta_l\}_{l=0}^{L} \) be an associated smooth partition of unity, such that \( \text{supp}(\eta_0) \subset \mathbb{R}^2 - \Sigma \), then \( \int \eta_0 \varphi d\mu_{i,j} = 0 \) for all \( i, j \) since \( (\mu_{i,j}) \) is supported on \( \Sigma \). For any other \( l \geq 1 \) either
claim #2 or claim #3 applies to $\eta \varphi$ and therefore

$$\int \eta \varphi \, d\mu_{i,j} = \int \eta \varphi \, \sigma_{i,j} \, d\mathcal{H}^1 \setminus \Sigma$$

for all $i, j$ and $l \geq 1$.

This proves the lemma since $\varphi = \sum_{l=0}^{L} \eta_l \varphi$.

The following is a partial result of our theorem for this particular situation. This time we will prove by hand that the second derivative of $f$ is a stationary 1-varifold.

**Lemma 26.** Let $f = \sup_{i=1,\ldots,N} g_i$ be the supremum over finitely many hyperplanes i.e. $g_i(x) = a_i \cdot x + b_i$ with $a_i \in \mathbb{R}^2$, $b_i \in \mathbb{R}$ and $a_i \neq a_j$ for $i \neq j$. Its second derivative, the Radon measure $(\mu_{i,j})$, satisfies the stationary condition (1.3) and is therefore the second moment of a stationary 1-varifold.

**Proof.** With the same argument that concluded the last proof we can assume without loss of generality that the support of the vector fields $\psi \in C^1_c(\mathbb{R}^2; \mathbb{R}^2)$ are contained in an open set $U$ with either $U \subset \mathbb{R}^2 - \Sigma$ or the situation of claim #2 and #3 respectively applies.

**Case 1:** This case is trivial since the measure $(\mu_{i,j})$ is supported on $\Sigma$:

$$\sum_{i,j=1}^{2} \left( \int \frac{\partial \psi_i}{\partial x_i} d\mu_{j,j} - \int \frac{\partial \psi_i}{\partial x_j} d\mu_{i,j} \right) = 0 \quad \forall \psi \in C^2(\mathbb{R}^2; \mathbb{R}^2) \text{ with } \text{supp}(\psi) \subset \mathbb{R}^2 - \Sigma.$$

**Case 2:** Let $\psi$ be a $C^1$-vector field with $\text{supp}(\psi) \subset U$ and the situation of claim #2 applies i.e. $f = \max\{g_2, g_2\}$ on $U$. As usual $n = \frac{a_2 - a_1}{|a_2 - a_1|}$ and $t = \left( \begin{array}{c} 0 \\ -1 \end{array} \right) n$. $U \cap \Sigma$ is obviously a $C^1$ submanifold with chart $\phi : I \rightarrow U \cap \Sigma$ defined by $s \mapsto s t$

on an open subset $I \subset \mathbb{R}$. Apply the area formula or rather integration over submanifolds due to $(\mu_{i,j}) = (\sigma_{i,j}) \mathcal{H}^1 \setminus \Sigma$ to the stationary condition in form of (1.7)

$$\sum_{i,j=1}^{2} \left( \int_{U \cap \Sigma} \frac{\partial \psi_i}{\partial x_i} \sigma_{j,j} - \frac{\partial \psi_i}{\partial x_j} \sigma_{i,j} \, d\mathcal{H}^1 \right) = \int_{U \cap \Sigma} (\text{div}(\psi) - \text{trace}(D\psi n \otimes n)) |a_2 - a_1| d\mathcal{H}^1 = \int_{I} (\text{div}(\psi) - \text{trace}(D\psi n \otimes n))(s t) |a_2 - a_1| ds.$$

Choose $\{t, n\}$ as orthonormal basis to evaluate div and trace so that

$$(\text{div}(\psi) - \text{trace}(D\psi n \otimes n))(s t) = t \cdot D\psi(s t)t = \frac{d}{ds} ((t \cdot \psi)(s t)).$$
Together with $\psi(s t) = 0$ for all $s \in \partial I$, because $\text{supp}(\varphi) \subset U$, we have
\[
\int_I (\text{div}(\psi) - \text{trace}(D\psi n \otimes n))(s t) \left| a_2 - a_1 \right| ds = \int_I \frac{d}{ds} \left( (t \cdot \psi)(s t) \right) \left| a_2 - a_1 \right| ds = 0.
\]

**Case 3:** As in claim #3 of the previous lemma let $x \in S$ and $r > 0$ be small enough. Such that $f$ looks on $B(x, r)$ as sketched in figure 4.2. Without loss of generality we can assume that $x = 0$. Let $\psi \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}(\psi) \subset B(0, r)$ arbitrary. We choose the following notation $n_i := \frac{a_{i+1} - a_i}{|a_{i+1} - a_i|}$, $t_i = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) n_i$ with the convention that $g_{M+1} = g_1$.

At first we evaluate the situation sketched in figure 4.3. As before $E_i \subset \mathbb{R}^2$ denotes the segment of the circle containing just $\{g_{i+1} = g_i\} \cap B(0, r)$ of $B(0, r) \cap \Sigma$. Set $E_i^\epsilon := E_i - B(0, \epsilon)$. As in the previous case we use the area formula and the local chart $\phi_i : [0, r) \to (U \cap \Sigma) \cap E_i, f s \mapsto st_i$ to evaluate
\[
\sum_{k,j=1}^{2} \left( \int_{E_i^\epsilon \cap (U \cap \Sigma)} \frac{\partial \psi_k}{\partial x_k} \sigma_{j,i} - \frac{\partial \psi_k}{\partial x_j} \sigma_{k,i} \, d\mathcal{H}^1 \right) = \int_{(U \cap \Sigma) \cap E_i^\epsilon} (\text{div}(\psi) - \text{trace}(D\psi n \otimes n))(s t_i) \left| a_{i+1} - a_i \right| d\mathcal{H}^1
\]
\[
= \int_{E_i^\epsilon \cap (U \cap \Sigma)} (\text{div}(\psi) - \text{trace}(D\psi n \otimes n))(s t_i) \left| a_{i+1} - a_i \right| ds
\]
\[
= \left| a_{i+1} - a_i \right| (t_i \cdot \psi(r t_i) - t_i \cdot \psi(\epsilon t_i))
\]
\[
= -((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) (a_{i+1} - a_i)) \cdot \psi(\epsilon t_i)
\]

All together we obtain
\[
\sum_{k,j=1}^{2} \left( \int_{U \cap \Sigma} \frac{\partial \psi_k}{\partial x_k} \sigma_{j,i} - \frac{\partial \psi_k}{\partial x_j} \sigma_{k,i} \, d\mathcal{H}^1 \right)
\]
\[
= \sum_{k,j=1}^{2} \left( \int_{B(0, \epsilon) \cap \Sigma} \frac{\partial \psi_k}{\partial x_k} \sigma_{j,i} - \frac{\partial \psi_k}{\partial x_j} \sigma_{k,i} \, d\mathcal{H}^1 \right)
\]
\[
+ \sum_{i=1}^{M} \sum_{k,l=1}^{2} \left( \int_{E_i^\epsilon \cap (U \cap \Sigma)} \frac{\partial \psi_k}{\partial x_k} \sigma_{l,i} - \frac{\partial \psi_k}{\partial x_l} \sigma_{k,i} \, d\mathcal{H}^1 \right)
\]
\[
= \int_{B(0, \epsilon) \cap \Sigma} (\text{div} \psi - \text{trace}(D\psi \sigma)) \, d\mathcal{H}^1 - \sum_{i=1}^{M} ((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) (a_{i+1} - a_i)) \cdot \psi(\epsilon t_i).
\]

Passing to the limit $\epsilon \to 0$, taking into account that $
\sum_{i=1}^{M} (a_{i+1} - a_i) = 0$, since $a_{M+1} = a_1$, we get
\[
\int_{U \cap \Sigma} (\text{div} \psi - \text{trace}(D\psi \sigma)) \, d\mathcal{H}^1 = -\sum_{i=1}^{M} ((\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}) (a_{i+1} - a_i)) \cdot \psi(0) = 0.
\]
4.1.2 2. Proof of a part of the representation theorem

Now we are going to use the previous results about maxima over finitely many hyperplanes to prove the following special case of theorem 6.

Lemma 27. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be convex, its second derivative $D^2 f$ considered as a $\mathbb{R}^{2\times 2}$-valued Radon measure $(\mu_{i,j})$ satisfies the stationary condition (1.3) i.e. it is the second moment of a stationary 1-varifold.

Proof. For every $n \in \mathbb{N}$ there is a finite collection of points $\{x^n_1, \ldots, x^n_{M_n}\} \subset \overline{B(0,n)}$ such that $\overline{B(0,n)} \subset \bigcup_{j=1}^{M_n} B(x^n_j, \frac{1}{n})$. Apply the classical theorem about supporting hyperplanes, e.g. [RS80] Theorem V.4 or [KP99] Theorem 6.2.5, to the convex set $\text{epi}(f) := \{(x,y) \in \mathbb{R}^2 \times \mathbb{R} : y \geq f(x)\} \subset \mathbb{R}^3$ to every $(x^n_j, f(x^n_j))$; hence there is an affine function, a hyperplane, $g^{(n)}_j$ such that

$$g^{(n)}_j(x) \leq f(x) \quad \forall x \in \mathbb{R}^2$$

$$g^{(n)}_j(x^n_j) = f(x^n_j).$$

We define the convex function $f_n : \mathbb{R}^2 \to \mathbb{R}$ by

$$f_n := \sup_{j=1,\ldots,M_n} g^{(n)}_j.$$

Then $f_n(x) \leq f(x) \quad \forall x$ and $f_n(x^n_j) = f(x^n_j)$ for all $j = 1, \ldots, M_n$. Thus $f_n \to f$ for $n \to \infty$ uniformly on every convex set. As final result of the previous investigation about finitely many hyperplanes we obtained in lemma 26 that the second derivative of $f_n$ the $\mathbb{R}^{2\times 2}$-valued Radon measure $(\mu^{(n)}_{i,j})$ satisfies the stationary condition (1.3). The claim follows if we can show that $f_n \to f$ implies $\mu^{(n)}_{i,j} \to \mu_{i,j}$ weakly i.e. in the sense of Radon measures. To see that fix any $\varphi \in C_c(\mathbb{R}^2; \mathbb{R})$. Choose $r > 0$ sufficient large such that $\text{supp}(\varphi) \subset B(0,r)$. Remark 7 provides for all $n$.

$$|\mu^{(n)}_{i,j}|(B(0,r)) \leq 2 \sup_{x \in B(0,r+1)} |f_n| 2\pi r.$$

The same inequality holds for $\mu_{i,j}$ itself. Let $\epsilon > 0$, choose $\phi \in C_c(\mathbb{R}^2; \mathbb{R})$ such that $\text{supp}(\phi) \subset B(0,r)$ and

$$|\varphi - \phi| < \frac{\epsilon}{8\pi r \sup_{x \in B(0,r+1)} |f|}.$$
then
\[
\lim_{n \to \infty} \left| \int \varphi \, d\mu_{i,j}^{(n)} - \int \varphi \, d\mu_{i,j} \right| \\
\leq \lim_{n \to \infty} \left( \|\varphi - \phi\|_\infty |\mathcal{B}(0,r)| + \|\varphi - \phi\|_\infty |\mu_{i,j}^{(n)}(B(0,r))| + \left| \int (f_n - f) \frac{\partial^2 \phi}{\partial x_i \partial x_j} \, dx \right| \right) \\
< \lim_{n \to \infty} \left( \frac{\epsilon}{\sup_{x \in B(0,\rho+1)} |f_n|} \sup_{x \in B(0,\rho+1)} |f_n| + \frac{\epsilon}{2} + \left| \int (f_n - f) \frac{\partial^2 \phi}{\partial x_i \partial x_j} \, dx \right| \right) = \epsilon.
\]

Since \( \epsilon > 0 \) has been arbitrary we conclude \( \mu_{i,j}^{(n)} \rightharpoonup \mu_{i,j} \) so that for all compactly supported \( C^1 \)-vector fields \( \psi : \mathbb{R}^2 \to \mathbb{R}^2 \)

\[
\sum_{i,j=1}^{2} \left( \int \frac{\partial \psi_i}{\partial x_i} \, d\mu_{i,j} - \int \frac{\partial \psi_i}{\partial x_j} \, d\mu_{i,j} \right) = \lim_{n \to \infty} \sum_{i,j=1}^{2} \left( \int \frac{\partial \psi_i}{\partial x_i} \, d\mu_{i,j}^{(n)} - \int \frac{\partial \psi_i}{\partial x_j} \, d\mu_{i,j}^{(n)} \right) = 0.
\]

\( \square \)

### 4.2 Proof of the corollary about ”tangent cones”

We will conclude this chapter with the proof of the corollary about ”tangent cones”.

**Proof of Corollary.** Without loss of generality we can assume \( x = 0 \) and \( f(0) = 0 \), otherwise consider \( z \mapsto f(x + z) - f(x) \). We write \( \mu \) for the mass of the \( n \)-varifold i.e. \( \mu(A) = \mu_{1,1}(A) + \cdots + \mu_{n+1,n+1}(A) \) for all Borel sets \( A \subset \mathbb{R}^{n+1} \).

**Claim #1.** The \( n \)-dimensional upper density \( \theta_n(\mu,0) \) is finite.

**Proof of Claim #1.** As before Fix \( 0 < \rho_0 \) such that \( \overline{B(0,2\rho_0)} \subset U \). Apply corollary 22 to obtain for any \( 0 < \delta < \rho_0 \) and \( 0 < \rho < \rho_0 \)

\[
\mu(B(0,\rho)) \leq \frac{1}{\delta} \sup_{x \in \partial B(0,\rho)} \left| f(y) - f(x) \right| \mathcal{H}^n(\partial B(0,\rho)).
\]

(4.14)

\( \mathcal{H}^n(\partial B(0,\rho)) = (n+1) w_{n+1} \rho^n \) so that for any \( 0 < \rho < \rho_0 \) and \( 0 < \delta < \rho_0 \)

\[
\theta_n(\mu,0) \leq \frac{(n+1) w_{n+1}}{w_n} \frac{1}{\delta} \sup_{x \in \partial B(0,\rho)} \left| f(y) - f(x) \right|.
\]

It is interesting to observe that this inequality implies as well that \( \theta_n(\mu,0) = 0 \) whenever \( f \) is differentiable in 0, since considering \( z \mapsto f(z) - \nabla f(0) \cdot z \) instead of \( f \) we can always assume that \( \nabla f(0) = 0 \) whenever it is differentiable in 0.

Naturally the classical argument based on the monotonicity formulae would work as well, compare [Sim83] lemma 40.5.
That convex functions have one-sided derivatives is easy to see: Let \( x \in U \) arbitrary for any \( y \in \mathbb{R}^{n+1} \) and \( 0 < s \leq t < \delta \) with \( \delta \) small enough such that \( B(x, \delta|y|) \subset U \) we have \( \alpha := \frac{1}{2} \in (0, 1] \) and

\[
f(x + sy) = f(\alpha x + \alpha (x + ty)) \leq (1 - \alpha) f(x) + \alpha f(x + ty).
\]

Subtracting \( f(x) \) and dividing by \( s \) leads to

\[
s^{-1} (f(x + sy) - f(x)) \leq t^{-1} (f(x + ty) - f(x)).
\]

Thus \( t \in (0, \delta] \mapsto t^{-1} (f(x + ty) - f(x)) \) is non-decreasing. This implies that \( g \) given in (1.9) is well-defined. Furthermore it is positively homogeneous i.e. \( g(\lambda x) = \lambda g(x) \) for \( \lambda > 0 \) and \( g(x) \leq f(x) \forall x \in U \). As a limit of convex functions \( g \) is itself convex.

Therefore we have for any \( 0 < \rho < \rho_0 \) and \( 0 < \delta < \rho_0 \) as above with \( \nu \) the mass of the \( n \)-varifold represented by \( g \)

\[
\rho^{-n} \nu(B(0, \rho)) \leq \frac{(n + 1) w_{n+1}}{\delta} \sup_{x \in \partial B(0, \rho)} |g(x)| \leq \frac{2(n + 1) w_{n+1}}{\delta} \sup_{x \in B(0, \rho + \delta)} |f(x)|.
\]

**Claim #2.** \((\nu_{i,j})\) is a cone, in the sense the \( n \)-varifold represented by \((\nu_{i,j})\) is invariant under the image of all homotheties \( x \mapsto \lambda^{-1}x, \lambda > 0 \).

**Proof of Claim #2.** We have to show that for any \( \lambda > 0, \varphi \in C_c(\mathbb{R}^{n+1}; \mathbb{R}) \) and \( i, j = 1, \ldots, n \)

\[
\lambda^{-n} \int \varphi(\lambda^{-1} x) \, d\nu_{i,j}(x) = \int \varphi \, d\nu_{i,j}.
\]

One proves (4.15) for all \( \varphi \in C^2(\mathbb{R}^{n+1}; \mathbb{R}) \), and then it extends to \( C_c(\mathbb{R}^{n+1}; \mathbb{R}) \) by density. So let \( \varphi \in C^2_c(\mathbb{R}^{n+1}; \mathbb{R}) \) then

\[
\lambda^{-n} \int \varphi(\lambda^{-1} x) \, d\nu_{i,j}(x) = \lambda^{-n-2} \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j} (\lambda^{-1} x) g(x) \, dx.
\]

Performing a change of variables \( z = \lambda^{-1} x \) and positively homogeneity of \( g \)

\[
= \lambda^{-1} \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j} (z) g(\lambda z) \, dz = \lambda^{-1} \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j} (z) g(z) \, dz
\]

\[
= \int \varphi \, d\nu_{i,j}.
\]

**Claim #3.** \( \mu_{i,j,0,\lambda} \rightarrow \nu_{i,j} \) in the sense of Radon measures as \( \lambda \rightarrow 0 \).

**Proof of Claim #3.** Let \( \varphi \in C^3_c(\mathbb{R}^{n+1}; \mathbb{R}) \). Choose \( R > 0 \) large and \( \lambda_0 > 0 \) sufficient
small such that \( \text{supp}(\varphi) \subset B(0, R) \) and \( B(0, \lambda_0 R) \subset U \). Then for any \( 0 < \lambda < \lambda_0 \) performing a change of variable as above

\[
\int \varphi \, d\mu_{i,j,0,\lambda} = \lambda^{-n} \int \varphi(\lambda^{-1} x) \, d\mu_{i,j}(x)
\]

\[
= \lambda^{-n-2} \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(\lambda^{-1} x) \, f(x) \, dx = \lambda^{-1} \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(z) \, f(\lambda z) \, dz
\]

\[
= \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(z) \frac{f(\lambda z)}{\lambda} \, dz.
\]

Remember \( \lambda \mapsto \lambda^{-1} f(\lambda z) \) is non-decreasing for any \( z \in \mathbb{R}^{n+1} \) so that we can take the limit \( \lambda \to 0 \) under the integral and obtain

\[
\lim_{\lambda \to 0} \int \varphi \, d\mu_{i,j,0,\lambda} = \int \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(z) \, g(z) \, dz = \int \varphi \, d\nu_{i,j}. \tag{4.16}
\]

As established in claim \#1 and subsequently \( \mu_{0,\lambda}(B(0,R)), \nu(B(0,R)) \) are uniformly bounded so that the convergence in (4.16) extends to \( C_c(\mathbb{R}^{n+1};\mathbb{R}) \) proving the corollary. \( \Box \)
Bibliography


