Banach space valued ultradistributions and applications to abstract Cauchy problems

P. C. Kunstmann

Abstract

We define the convolution of Banach space valued ultradistributions in the sense of Braun, Meise, and Taylor. We then treat abstract Cauchy problems in Banach spaces as convolution equations and give a characterization of those problems that have ultradistributional fundamental solutions. Our characterization extends in the Beurling case a result due to H.A. Emamirad. We apply our result to differential operators in Banach spaces of ultradifferentiable functions with different (e.g. ultradifferential) boundary conditions.

Introduction

A few years ago, Braun, Meise and Taylor ([2]) introduced a new frame for ultradistributions which extended the definition of Beurling-Björck and was shown to be equivalent (in the sense of [4]) to the ultradistribution theory of Komatsu (with conditions (M.1) and (M.3')) and to the theory of Cioränescu and Zsidó ([4]) while actually refining the latter (see [2], Lemma 8.6) and extending it to the n-dimensional case.

In this paper we apply this theory of ultradistributions to the study of abstract Cauchy problems in Banach spaces of the following form. Suppose we have two Banach spaces E, D and a finite family of operators (A_j) in $L(D, E)$. We want to solve the equation

$$\sum_j A_j u^{(j)}(t) = f(t),$$

where f is a given ultradistribution with support bounded from below and values in E and we search for a solution u which is an ultradistribution with values in D and support contained in $[\inf \operatorname{supp} f, \infty[$.

If we denote $P := \sum_j \delta^{(j)} \otimes A_j$ and P has what is called a *fundamental solution* G then the unique solution of this problem is $G * f$.

Thus, if we look for fundamental solutions in the spaces $\mathscr{D}'_*(L(E, D)) := L_b(\mathscr{D}_*, L(E, D))$ where $* \in \{(\omega), \{\omega\}\}$ and ω is a weight function in the sense of [2], then – in order to get the solution – we need a convolution for Banach space valued ultradistributions. Notice that we can not apply the results of [8] since we do not deal with spaces of distributions here.

The aim of this paper is twofold. First we establish a convolution for spaces of Banach space valued ultradistributions with support bounded from below. The main proceeding is as in [8]. Since the values are in a Banach space and we have nice properties of the scalar-valued convolution we can use the π-tensor product throughout and thus the arguments are considerably simplified. We then characterize those P of the form above that possess a fundamental solution in the spaces of ultradistributions under consideration and give some applications of this characterization to concrete problems.

The paper is organized as follows. The first section recalls some of the definitions from [2] and introduces the spaces $\mathscr{D}_{*,-}$, $\mathscr{D}'_{*,+}$ we need to define convolution, as well as spaces of Banach space valued ultradistributions. In section 2 we study the topological properties of those spaces and in section 3 we define the convolution and present its continuity properties. Section 4 covers Paley-Wiener-Theorems for $\mathscr{E}'_*(X)$ (and is a refinement of results in [2]). In section 5 we characterize those P of the form above that possess fundamental solutions in $\mathscr{D}'_{*,+}(L(E,D))$ through properties of the Laplace transform $\mathscr{L}P$ of P. In the Beurling case our characterization extends a result due to H.A. Emamirad ([5]). In the Roumieu case the characterization seems to be new. In the last section we treat some differential operators in a Banach space of ultradifferentiable functions under different boundary conditions involving differential and ultradifferential operators and show how the characterization theorem can be applied.

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1 Ultradifferentiable functions and ultradistributions

We will use the following definition from [2].

Definition 1.1 Let $\omega : [0,\infty[\to [0,\infty[$ be a continuous increasing function with $\omega|_{[0,1]} = 0$. It is called a *weight function* if it satisfies

$$(\alpha)\ \text{there exists } K \geq 1 \text{ with } \omega(2t) \leq K(1+\omega(t)) \text{ for all } t \geq 0,$$

$$(\beta)\ \int_1^\infty \frac{\omega(t)}{1+t^2}\,dt > \infty,$$

$$(\gamma)\ \lim_{t\to\infty} \frac{\log(1+t)}{\omega(t)} = 0,$$

$$(\delta)\ \varphi : [0,\infty[\to [0,\infty[, t \mapsto \omega(e^t) \text{ is convex.}$$

The *convex conjugated function* φ^* *of* φ is given by $\varphi^*(s) = \sup_{t\geq 0}(st - \varphi(t))$ for all $s \geq 0$. Two weight functions ω and σ are *equivalent* if $\omega = O(\sigma)$ and $\sigma = O(\omega)$.

We fix a weight function ω. Our first lemma states that ω can be regularized up to equivalence.

Lemma 1.2 *There is a weight function σ which is equivalent to ω and infinitely differentiable on* $]1,\infty[$.

Proof. Choose a $\rho \in C^\infty$ with $\operatorname{supp}\rho \subset [-1,0]$, $\rho \geq 0$ and $\int \rho = 1$. Extend φ to $\mathbb{R}$ by letting $\varphi(t) = 0$ for $t < 0$. Set $\psi := \varphi * \rho$. Then ψ is infinitely differentiable, convex, and satisfies $\varphi(s) \leq \psi(s) \leq \varphi(s+1)$. Hence σ given by $\sigma(t) := 0$ for $t \in [0,1[$ and $\sigma(t) := \psi(\log t) - \psi(0)$ has the desired properties. □

As in [2] we define for all compact sets $K \subset \mathbb{R}$ and $h > 0$ the Banach space

$$\mathscr{D}^h_\omega(K) := \{f \in \mathscr{D}(K) : |f|_h := \int_{\mathbb{R}} |\hat{f}(t)| \exp(h\omega(t))\,dt < \infty\}$$

and let

$$\mathscr{D}_{\{\omega\}}(K) := \operatorname{ind}_h \mathscr{D}^h_\omega(K) \qquad \mathscr{D}_{(\omega)}(K) := \operatorname{proj}_h \mathscr{D}^h_\omega(K)$$

as well as

$$\mathscr{D}_* := \operatorname{ind}_K \mathscr{D}_*(K).$$

We further define

$$\|f\|_{b,A} := \sup_{k\in\mathbb{N}_0} \sup_{t\in A} |f^{(k)}(t)| \exp\left(-b\varphi^*(k/b)\right)$$

for all $f \in C^\infty$, $b > 0$ and $A \subset \mathbb{R}$, and write $\|f\|_b$ for $\|f\|_{b,\mathbb{R}}$. Then Lemma 3.3 in [2] shows

$$\|f\|_b \leq \frac{1}{2\pi}|f|_b, \qquad |f|_c \leq m(\operatorname{supp} f)\|f\|_b \cdot \operatorname{const}(b,c), \quad f \in \mathscr{D},$$

for all $b > 0$ and $c - b/L \geq 2$ where L is an absolut constant and m denotes Lebesgue measure.

For any Banach space X we define

$$\mathscr{D}'_*(X) := L_b(\mathscr{D}_*, X)$$

where the subscript $_b$ indicates that these spaces are supplied with the topology of uniform convergence on the bounded subsets of the underlying test function space. For $* \in \{(\omega), \{\omega\}\}$ and any subset $B \subset \mathbb{R}$ we let

$$\mathscr{E}_{*,B} := \{f \in \mathscr{E}_* : \operatorname{supp} f \subset B\},$$

supplied with the induced topology, and define

$$\mathscr{D}_{*,-} := \operatorname{ind}_{n\to\infty} \mathscr{E}_{*,]-\infty,n]}.$$

We further define

$$\mathscr{D}'_{*,+} := (\mathscr{D}_{*,-})'_b \quad \text{and} \quad \mathscr{D}'_{*,+}(X) := L_b(\mathscr{D}_{*,-}, X)$$

for any Banach space X. For $a \in \mathbb{R}$ and any Banach space X let

$$\mathscr{D}'_{*,[a,\infty[}(X) := \{T \in \mathscr{D}'_*(X) : \operatorname{supp} T \subset [a,\infty[\},$$

supplied with the topology induced by $\mathscr{D}'_*(X)$. $\mathscr{D}'_{*,[a,\infty[}(X)$ is a closed subspace of $\mathscr{D}'_*(X)$.

As a set $\mathscr{D}'_{*,+}(X)$ is in a canonical way the union of all $\mathscr{D}'_{*,[a,\infty[}(X)$, $a \in \mathbb{R}$. The topology on $\mathscr{D}'_{*,[a,\infty[}(X)$ that is inherited from $\mathscr{D}'_{*,+}(X)$ coincides with the one inherited from $\mathscr{D}'_*(X)$.

2 Topological properties of test function and distribution spaces

We obtain the topological properties of $\mathscr{D}'_{*,+}$ we need for the definition of the convolution by representing these spaces as complemented subspaces in sequence spaces. We want to remark that from the definition it is already clear that $\mathscr{D}_{(\omega),-}$ is a nuclear LF-space.

Definition 2.1 If E and F are locally convex spaces we use the notation $E \prec_{cs} F$ if E is isomorphic to a complemented subspace of F and $E \sim_{cs} F$ if $E \prec_{cs} F$ and $F \prec_{cs} E$.

For every locally convex space E we define

$$\psi(E) := \operatorname{ind}_{n\to\infty} \left(\prod_{k=-\infty}^{n} E \right).$$

As is easily seen we have

Lemma 2.2 *$\psi(E) = E^{\mathbb{N}} \times \bigoplus_{\mathbb{N}} E$ for every locally convex space E.*

Corollary 2.3 *We have $\psi(E)'_b \cong \psi(E'_b)$. Moreover, $\psi(E)$ is reflexive (complete, nuclear, ultrabornological) if and only if E is. If $E \prec_{cs} F$ then $E'_b \prec_{cs} F'_b$ and E is reflexive (complete, nuclear, ultrabornological) if F is.*

Theorem 2.4 *Letting $\alpha(\omega) := (\omega(j))_{j\in\mathbb{N}}$ we have*

$$\mathscr{D}_{(\omega),-} \sim_{cs} \psi(\Lambda_\infty(\alpha(\omega))) \quad \textit{and} \quad \mathscr{D}_{\{\omega\},-} \sim_{cs} \psi(\Lambda_1(\alpha(\omega))'_b),$$
$$\mathscr{E}_{(\omega),]-\infty,0]} \cong (\Lambda_\infty(\alpha(\omega)))^{\mathbb{N}} \quad \textit{and} \quad \mathscr{E}_{\{\omega\},]-\infty,0]} \cong (\Lambda_1(\alpha(\omega))'_b)^{\mathbb{N}},$$
$$\mathscr{D}'_{(\omega),[0,\infty[} \cong (\Lambda_\infty(\alpha(\omega))'_b)^{\mathbb{N}} \quad \textit{and} \quad \mathscr{D}'_{\{\omega\},[0,\infty[} \cong (\Lambda_1(\alpha(\omega)))^{\mathbb{N}}.$$

Proof. We proceed as in the proof of [10], Theorem 5.1 and 5.2. For the sake of completeness we repeat the arguments in the first two cases. Let $K_j := [(j-1)\pi, (j+1)\pi]$ and $P_j : \mathscr{D}_*(K_j) \to \mathscr{E}_{*,2\pi}$ denote the extension to 2π-periodical functions for every $j \in \mathbb{Z}$. Choose a partition of unity (χ_j) subordinated to $int(K_j)$ and functions $\psi_j \in \mathscr{D}_*(K_j)$ with $\psi_j \chi_j = \chi_j$. Then the mappings

$$\Phi : \mathscr{D}_{*,-} \to \psi(\mathscr{E}_{*,2\pi}), f \mapsto (P_j(\chi_j f))_{j\in\mathbb{Z}} \text{ and } \Psi : \psi(\mathscr{E}_{*,2\pi}) \to \mathscr{D}_{*,-}, (g_j)_{j\in\mathbb{Z}} \mapsto \sum_{j\in\mathbb{Z}} \psi_j g_j$$

are continuous (note that in the definition of Ψ the sum is locally finite) and for every $f \in \mathscr{D}_{*,-}$ we have

$$\Psi \circ \Phi(f) = \sum_{j\in\mathbb{Z}} \psi_j \cdot P_j(\chi_j f) = \sum_{j\in\mathbb{Z}} \psi_j \chi_j f = \sum_{j\in\mathbb{Z}} \chi_j f = f.$$

Hence $\mathscr{D}_{*,-} \prec_{cs} \psi(\mathscr{E}_{*,2\pi})$. For the reverse let $K_j := [(8j-2)\pi, (8j+2)\pi]$ and denote by Q_j the mapping $f \mapsto \sum_{\nu\in\mathbb{Z}} f(\cdot - 2\nu\pi)$ from $\mathscr{D}_*(K_j)$ to $\mathscr{E}_{*,2\pi}$. Since the sum is locally finite each Q_j is continuous. Moreover we can find χ_j with support in $int(K_j)$ such that $Q_j \chi_j = 1$. Choose $\psi_j \in \mathscr{D}_*(K_j)$ with $\psi_j \chi_j = \chi_j$ and define

$$\Psi : \psi(\mathscr{E}_{*,2\pi}) \to \mathscr{D}_{*,-}, (g_j)_{j\in\mathbb{Z}} \mapsto \sum_{j\in\mathbb{Z}} \chi_j g_j \text{ and } \Phi : \mathscr{D}_{*,-} \to \psi(\mathscr{E}_{*,2\pi}), f \mapsto (Q_j(\psi_j f))_{j\in\mathbb{Z}}.$$

Then both Φ and Ψ are continuous and for every $(g_j) \in \psi(\mathscr{E}_{*,2\pi})$ we have

$$\Psi \circ \Phi(g_j)_j = (Q_j(\psi_j \sum_\nu \chi_\nu g_\nu))_j = (Q_j(\psi_j \chi_j g_j))_j = (Q_j(\chi_j g_j))_j = (Q_j(\chi_j) \cdot g_j)_j = (g_j)_j.$$

Hence $\psi(\mathscr{E}_{*,2\pi})$ is isomorphic to a complemented subspace of $\mathscr{D}_{*,-}$. Using now the representation of $\mathscr{E}_{*,2\pi}$ and its dual from [2] we finished. The remaining assertions are proved in a similar way. □

Remark 2.5 Since $\psi(\psi) \ncong \psi$ we can not use Pelczynski's trick (as it is done in [10] for the sequence spaces ω, ϕ, s) to conclude that $\sim_{cs}$ can be replaced by $\cong$ in the theorem above.

Corollary 2.6 *We have*

$$\mathscr{D}'_{(\omega),+} \sim_{cs} \psi(\Lambda_\infty(\alpha(\omega))'_b) \quad \textit{and} \quad \mathscr{D}'_{\{\omega\},+} \sim_{cs} \psi(\Lambda_1(\alpha(\omega))).$$

In particular, these spaces are reflexive, nuclear, complete and ultrabornological.

Now using

Lemma 2.7 *Each bounded subset of $\mathscr{D}'_{*,+}(X)$ is contained in some $\mathscr{D}'_{*,[a,\infty[}(X)$, $a \in \mathbb{R}$.*

Proof. If this is not the case there is a bounded subset $B \subset \mathscr{D}'_{*,+}(X)$ and a sequence (T_n) in B such that $\operatorname{supp} T_n \not\subset [-n, \infty[$ holds for each $n \in \mathbb{N}$. Hence we find for each $n \in \mathbb{N}$ a test function $\psi_n \in \mathscr{E}_{*,]-\infty,-n]}$ with $c_n := \|T_n(\psi_n)\| \neq 0$. Then $M := \{n\psi_n/c_n : n \in \mathbb{N}\}$ is a bounded subset of $\mathscr{D}_{*,-}$, but

$$\sup_{\psi\in M, T\in B} |T(\psi)| \geq \sup_{n\in\mathbb{N}} \left|\frac{n}{c_n} T(\psi_n)\right| = \infty$$

which contradicts the boundedness of B. □

for the case $X = \mathbb{C}$ we easily prove

Corollary 2.8 *We have topologically $\mathscr{D}'_{*,+} = \operatorname{ind}_n \mathscr{D}'_{*,[-n,\infty[}$. In particular $\mathscr{D}'_{\{\omega\},+}$ is a (nuclear) LF-space.*

The proof of the follwing lemma is standard.

Lemma 2.9 *Let X be a Banach space and $\mathscr{H}$ be bornological. Then $L_b(\mathscr{H}, X)$ is complete.*

Proof. Let $(T_\alpha)_{\alpha\in A}$ be a Cauchy net in $L_b(\mathscr{H}, X)$. Since X is complete, (T_α) converges uniformly on bounded sets to a linear mapping $T : \mathscr{H} \to X$. Hence $T(B)$ is bounded for each bounded subset B of $\mathscr{H}$. Since $\mathscr{H}$ is bornological, T is continuous. Obviously the convergence $T_\alpha \to T$ holds in $L_b(\mathscr{H}, X)$. □

We conclude that $\mathscr{D}'_{*,+}(X)$ is complete. The next lemma is proved in the usual way.

Lemma 2.10 *$\mathscr{D}'_{*,+} \otimes X$ is a dense subspace of $\mathscr{D}'_{*,+}(X)$. The assertion remains true if we replace $\mathscr{D}'_{*,+}$ by $\mathscr{D}'_{*,[a,\infty[}$ for $a \in \mathbb{R}$.*

Proposition 2.11 *Let X be a Banach space. Then $\mathscr{D}'_{*,+}(X)$ is the complete hull of $\mathscr{D}'_{*,+} \otimes_\epsilon X = \mathscr{D}'_{*,+} \otimes_\pi X$. The assertion also holds for $\mathscr{D}'_{*,[a,\infty[}$ instead of $\mathscr{D}'_{*,+}$.*

Proof. The assertion follows from Lemma 2.10 and the nuclearity of $\mathscr{D}'_{*,+}$ and $\mathscr{D}'_{*,[a,\infty[}$, respectively, since $\otimes_\epsilon$ is the topology induced by $\mathscr{D}'_{*,+}(X)$. □

We close this section with the proposition that will be essential in our treatment of the convolution of Banach space valued ultradistributions.

Proposition 2.12 *For any Banach space X we have $\mathscr{D}'_{*,+}(X) = \mathrm{ind}_n \mathscr{D}'_{*,[-n,\infty[}(X)$.*

Proof. Using Corollary 2.8, Proposition 2.11 and the lemma from [6], p.47, we get

$$\begin{aligned} \mathrm{ind}_n \mathscr{D}'_{*,[-n,\infty[}(X) &= \mathrm{ind}_n (\mathscr{D}'_{*,[-n,\infty[} \widehat{\otimes}_\pi X) = \bigcup_n (\mathscr{D}'_{*,[-n,\infty[} \widehat{\otimes}_\pi X) \\ &= \bigcup_n (\mathscr{D}'_{*,[-n,\infty[}(X)) = \mathscr{D}'_{*,+}(X) \; \square \end{aligned}$$

Remark 2.13 By [9], Proposition 2, $\mathscr{D}'_{*,+}(X)$ is even ultrabornological and barrelled.

3 Convolution of vector valued ultradistributions

We first define convolution for scalar valued ultradistributions and start with a lemma on the approximation of the derivative.

Lemma 3.1 *For each C^∞-function f, all $t, h \in \mathbb{R}$, $b > 0$, and every interval $B \subset \mathbb{R}$ we have*

$$\|\frac{1}{h}(\tau_{-(t+h)} f - \tau_{-t} f) - \tau_{-t} f'\|_{b,B} \le |h| \exp(b\varphi^*(2/b)) \|f\|_{2b,B-t+[-|h|,|h|]}.$$

Proof. For all s and α we clearly have

$$|\frac{1}{h}(f^{(\alpha)}(s-(t+h)) - f^{(\alpha)}(s-t)) - f^{(\alpha+1)}(s-t)| \le |h| \sup_{r\in[s-(t+h),s-t]} |f^{(\alpha+2)}(r)|.$$

Hence we get the assertion from

$$2b\varphi^*(\frac{\alpha+2}{2b}) \le b\varphi^*(\alpha/b) + b\varphi^*(2/b)$$

which holds since φ^* is convex. □

Lemma 3.2 *Let $S \in \mathscr{D}'_{*,+}$. Then the linear mapping*

$$\mathscr{D}_{*,-} \to \mathscr{D}_{*,-}, f \mapsto (S * f^\vee)^\vee$$

is continuous where $f^\vee$ is defined by $f^\vee(t) := f(-t)$ for each $f : \mathbb{R} \to \mathbb{C}$. For all $a, c \in \mathbb{R}$ we have $\mathrm{supp}\,(S * f^\vee)^\vee \subset]-\infty, c-a]$ *if* $\mathrm{supp}\, S \subset [a, \infty[$ *and* $\mathrm{supp}\, f \subset]-\infty, c]$.

Proof. We find $a \in I\!R$ such that $\operatorname{supp} S \subset [a, \infty[$. Let $c \in I\!R$ and suppose $f \in \mathscr{E}_{*,]-\infty,c]}$. Then the mapping

$$I\!R \to \mathbb{C}, t \mapsto (S * f^\vee)^\vee(t) = S(\tau_{-t} f)$$

is infinitely differentiable by Lemma 3.1 with derivative $t \mapsto S(\tau_{-t} f')$ and has support in $]-\infty, c-a]$. We now treat the case $* = (\omega)$. Letting $k \in I\!N$ we find $m \in I\!N$ and $b > 0$ such that

$$|S(\psi)| \leq C \|\psi\|_{b,[-m,\infty[} \tag{1}$$

for all $\psi \in \mathscr{E}_{(\omega),]-\infty,c+k]}$. Letting $a > 0$ and using

$$(a+b)\varphi^*(\frac{\alpha+\beta}{a+b}) \leq (a+b)\left(\frac{a}{a+b}\varphi^*(\frac{\alpha}{a}) + \frac{b}{a+b}\varphi^*(\frac{\beta}{b})\right) = a\varphi^*(\frac{\alpha}{a}) + b\varphi^*(\frac{\beta}{b}) \tag{2}$$

we get

$$\|(S * f^\vee)^\vee\|_{a,[-k,\infty[} \leq C\|f\|_{a+b,[-k-m,\infty[}. \tag{3}$$

Hence we proved the first part of the lemma in the Beurling case. In the Roumieu case we fix $k \in I\!N$ and find $m \in I\!N$ such that (1) holds for all $b > 0$ and $\psi \in \mathscr{E}_{\{\omega\},]-\infty,c+k]}$. If d is positive then we set $a := b := d/2$ and (3) implies the first assertion in the Roumieu case. The second part is clear. □

Corollary 3.3 *Let $S, T \in \mathscr{D}'_{*,+}$. Then the mapping $S * T : f \mapsto S((T * f^\vee)^\vee)$ belongs to $\mathscr{D}'_{*,+}$. For all $a, b \in I\!R$ we have $\operatorname{supp} S * T \subset [a+b, \infty[$ if $\operatorname{supp} S \subset [a, \infty[$ and $\operatorname{supp} T \subset [b, \infty[$.*

We show the continuity properties we will need to establish the convolution of vector valued ultradistributions.

Proposition 3.4 *For all $a, b \in I\!R$ the mapping $* : \mathscr{D}'_{*,[a,\infty[} \times \mathscr{D}'_{*,[b,\infty[} \to \mathscr{D}'_{*,[a+b,\infty[}$ is continuous. The mapping $* : \mathscr{D}'_{*,+} \times \mathscr{D}'_{*,+} \to \mathscr{D}'_{*,+}$ is hypo-continuous with respect to the bounded subsets of $\mathscr{D}'_{*,+}$.*

Proof. We first treat the Beurling case. Let $B \subset \mathscr{D}_{(\omega),-}$ be bounded. Then we find $c \in I\!R$ with $B \subset \mathscr{E}_{(\omega),]-\infty,c]}$. Now

$$B_d := \left\{\exp(-d\varphi^*(\alpha/d))\tau_{-t} f^{(\alpha)} : \alpha \in I\!N_0, f \in B, t \in [a-1, c-b]\right\}$$

is a bounded subset of $\mathscr{E}_{(\omega),]-\infty,c+1-a]}$ for each $d > 0$ which follows from

$$\|\exp(-d\varphi^*(\alpha/d))\tau_{-t} f^{(\alpha)}\|_{l,[-m,\infty[} \leq \|f\|_{d+l,[-m+a-1,\infty[}$$

(here we used (2)). We find a bounded subset $\tilde{B}$ of $\mathscr{E}_{(\omega),]-\infty,c+1-a]}$ and a sequence $(k_d)_{d \in I\!N}$ of positive numbers such that

$$B_{dL_\omega} \subset k_d \tilde{B},\ d \in I\!N.$$

The set $V := \tilde{B}^\circ \cap \mathscr{D}'_{(\omega),[b,\infty[}$ is a neighbourhood of zero in $\mathscr{D}'_{(\omega),[b,\infty[}$. Choose $\chi \in \mathscr{E}_{(\omega)}$ with $\chi|_{[a\infty[} = 1$ and $\chi|_{]-\infty,a-1]} = 0$. Then

$$M := \{\chi(S * f^\vee)^\vee : f \in B, S \in V\}$$

is a bounded subset of $\mathscr{E}_{]-\infty,c-b]}$ since we have for each $d \in I\!N_0$ by Lemma 4.1

$$\begin{aligned} \sup_{f \in B, S \in V, \alpha \geq 0} \|\chi(S * f^\vee)^\vee\|_d &\leq \exp(dL_\omega)\|\chi\|_{dL_\omega} \sup_{f \in B, S \in V} \|(S * f^\vee)^\vee\|_{dL_\omega,[a-1,c-b]} \\ &\leq \exp(dL_\omega)\|\chi\|_{dL_\omega} k_d. \end{aligned}$$

Hence $U := M^\circ \cap \mathscr{D}'_{(\omega),[a,\infty[}$ is a neighbourhood of zero in $\mathscr{D}'_{(\omega),[a,\infty[}$. For all $T \in U$, $S \in V$ and $f \in B$ we then have

$$|T * S(f)| = |T((S * f^\vee)^\vee)| = |T(\chi(S * f^\vee)^\vee)| \leq 1,$$

i. e. $*(U \times V) \subset B^\circ \cap \mathscr{D}'_{(\omega),[a+b,\infty[}$. Thus the first assertion is proved in the Beurling case. The second assertion follows from the barrelledness of $\mathscr{D}'_{(\omega),+}$.

In the Roumieu case the assertions follow by Theorem 2.4 and Corollary 2.8 once we showed the separate sequential continuity of $*$. Since $*$ is commutative which is shown by $S * T(f) = S_\xi \otimes T_\eta(f(\xi + \eta))$ it suffices to show the sequential continuity in the first variable which holds since by Lemma 3.2 the set $\{(S * f^\vee)^\vee : f \in B\}$ is bounded in $\mathscr{D}_{\{\omega\},-}$ for each bounded subset $B \subset \mathscr{D}_{\{\omega\},-}$. □

Proposition 3.5 *For all Banach spaces E, F there is a unique separately continuous bilinear mapping*

$$*_{\pi,E,F} : \mathscr{D}'_{*,+}(E) \times \mathscr{D}'_{*,+}(F) \to \mathscr{D}'_{*,+}(E \widehat{\otimes}_\pi F) \tag{4}$$

such that $T \otimes x *_{\pi,E,F} S \otimes y = T * S \otimes (x \otimes y)$ holds for all elementary tensors.

This mapping is moreover hypo-continuous with respect to bounded sets, and for all $a, b \in \mathbb{R}$ the restricted mapping

$$*_{\pi,E,F} : \mathscr{D}'_{*,[a,\infty[}(E) \times \mathscr{D}'_{*,[b,\infty[}(F) \to \mathscr{D}'_{*,[a+b,\infty[}(E \widehat{\otimes}_\pi F)$$

is even continuous.

Proof. We get uniqueness immediately from Proposition 2.11. Let $a, b \in \mathbb{R}$. Then by Proposition 3.4 the linearized mapping

$$\widehat{*} : \mathscr{D}'_{*,[a,\infty[} \widehat{\otimes}_\pi \mathscr{D}'_{*,[b,\infty[} \to \mathscr{D}'_{*,[a+b,\infty[}$$

is continuous. Hence

$$\widehat{*} \otimes I : (\mathscr{D}'_{*,[a,\infty[} \widehat{\otimes}_\pi \mathscr{D}'_{*,[b,\infty[}) \widehat{\otimes}_\pi (E \widehat{\otimes}_\pi F) \to \mathscr{D}'_{*,[a+b,\infty[} \widehat{\otimes}_\pi (E \widehat{\otimes}_\pi F)$$

is continuous where I denotes the identity on $E \widehat{\otimes}_\pi F$. Now there is a canonical isomorphism

$$\iota : (\mathscr{D}'_{*,[a,\infty[} \widehat{\otimes}_\pi \mathscr{D}'_{*,[b,\infty[}) \widehat{\otimes}_\pi (E \widehat{\otimes}_\pi F) \to (\mathscr{D}'_{*,[a,\infty[} \widehat{\otimes}_\pi E) \widehat{\otimes}_\pi (\mathscr{D}'_{*,[b,\infty[} \widehat{\otimes}_\pi F).$$

If we denote the canonical bilinear mapping of $(\mathscr{D}'_{*,[a,\infty[} \widehat{\otimes}_\pi E) \times (\mathscr{D}'_{*,[b,\infty[} \widehat{\otimes}_\pi F)$ in the π-tensor product of both spaces by q, then $*_{\pi,E,F,a,b} := \widehat{*} \otimes I \circ \iota \circ q$ is by Proposition 2.11 a continuous bilinear mapping

$$\mathscr{D}'_{*,[a,\infty[}(E) \times \mathscr{D}'_{*,[b,\infty[}(F) \to \mathscr{D}'_{*,[a+b,\infty[}(E \widehat{\otimes}_\pi F)$$

which coincides on elementary tensors with the mapping induced by

$$(T \otimes x, S \otimes y) \mapsto T * S \otimes (x \otimes y).$$

If we denote by $\mathscr{U}_c(X)$ the space $\mathscr{D}'_{*,[c,\infty[}(X)$ for $X \in \{E, F, E \widehat{\otimes}_\pi$ then $\mathscr{U}_c(X)$ is a topological subspace of $\mathscr{U}_{\tilde{c}}(X)$ for every pair $c > \tilde{c}$. Hence the union of all the maps $*_{\pi,E,F,a,b}$ is a well defined map $*_{\pi,E,F}$ as in (4) which is clearly bilinear.

We now show hypo-continuity. Let B be bounded and absolutely convex subset of $\mathscr{D}'_{*,+}(E)$. By Lemma 2.7 we find an integer m such that B is a bounded subset in $\mathscr{U}_m(E)$. By Proposition 2.12 and the lemma from [6], p. 47, we get $< B > \widehat{\otimes}_\pi \mathscr{D}'_{*,+}(F) = \mathrm{ind}_n(< B > \widehat{\otimes}_\pi \mathscr{U}_{-n}(F))$. Since the linearized mappings $\widehat{*}_{\pi,E,F} :< B > \widehat{\otimes}_\pi \mathscr{U}_{-n} \to \mathscr{D}'_{*,+}(E \widehat{\otimes}_\pi F)$ are continuous this implies that $\widehat{*}_{\pi,E,F} :< B > \widehat{\otimes}_\pi \mathscr{D}'_{*,+}(F) \to \mathscr{D}'_{*,+}(E \widehat{\otimes}_\pi F)$ is continuous. If B is a bounded subset of $\mathscr{D}'_{*,+}(F)$ we proceed in the same way. □

Corollary 3.6 *Let E, F, G be Banach spaces and $b : E \times F \to G$ be a continuous bilinear mapping. Then there is a unique separately continuous bilinear mapping*

$$*_b : \mathscr{D}'_{*,+}(E) \times \mathscr{D}'_{*,+}(F) \to \mathscr{D}'_{*,+}(G)$$

such that $S \otimes x *_b T \otimes y = (S * T) \otimes b(x, y)$ holds on elementary tensors. Moreover, this mapping is hypo-continuous with respect to bounded sets.

Proof. We get uniqueness as above. Obviously $*_b := *_{\pi,E,F} \circ I \otimes \widehat{b}$ has the desired properties where I denotes the identity on $\mathscr{D}'_{*,+}$. □

Since $\otimes_\pi$ is associative, convolution of Banach space valued ultradistributions is associative under obvious assumptions on the bilinear mappings involved.

4 Paley-Wiener-Theorems for ultradistributions with compact support

We characterize ultradistributions with compact support through properties of their Laplace transforms. This section just refines some of the statements in [2].

Throughout this section ω is a weight function and we fix $L_\omega > 0$ with $\omega(et) \leq L_\omega(1 + \omega(t))$ for all $t \geq 0$. We then have $\varphi(x + 1) \leq L_\omega(1 + \varphi(x))$ for all $x \geq 0$ and

$$y + L_\omega \varphi^*(y/L_\omega) \leq \varphi^*(y) + L_\omega \quad \text{for all } y \geq 0. \tag{5}$$

We first prove a lemma on the multiplication of ultradifferentiable functions.

Lemma 4.1 *Let $K \subset \mathbb{R}$ and $h > 0$. Then we have for all $f, g \in C^\infty$*

$$\|fg\|_{K,h} \leq \exp(hL_\omega)\|f\|_{K,hL_\omega}\|g\|_{K,hL_\omega}.$$

Proof. The proof follows that of [2], Proposition 4.4. Using (5) for $y = \alpha/h$ we clearly have for $l := hL_\omega$

$$\begin{aligned} |(fg)^{(\alpha)}(t)| &\leq 2^\alpha \max_{\beta=0,\dots,\alpha} exp(l\varphi * (\beta/l) + l\varphi^*((\alpha-\beta)/l))\|f\|_{K,l}\|g\|_{K,l} \\ &\leq exp(\alpha + l\varphi^*(\alpha/l))\|f\|_{K,l}\|g\|_{K,l} \\ &\leq exp(h(\varphi^*(\alpha/h) + L_\omega))\|f\|_{K,l}\|g\|_{K,l} \quad \square \end{aligned}$$

We now state the Paley-Wiener-Theorem for $\mathscr{E}'_{(\omega)}(X)$.

Theorem 4.2 *Let $K \subset \mathbb{R}$ be compact and X be a Banach space. Consider the following two assertions:*

(a) $T \in \mathscr{E}'_{(\omega)}(X)$ and $\operatorname{supp} T \subset K$.

(b) $F : \mathbb{C} \to X$ is entire and there is an $m > 0$ such that for all compact sets $L \subset \mathbb{R}$ with $K \subset int(L)$ there is a $C_L > 0$ such that

$$\|F(\lambda)\| \leq C_L \exp(H_L(-Re\,\lambda) + m\omega(|\lambda|)), \quad \lambda \in \mathbb{C}.$$

If (a) holds then (b) holds for $F := \mathscr{L}T$. If (b) holds then there is T with (a) and $\mathscr{L}T = F$.

Proof. Let (a) hold. Just like in [2], 7.1, we can show that if

$$\|T(f)\| \leq C\|f\|_{M,h}, \quad f \in \mathscr{E}_*, \tag{6}$$

for some compact set $M \subset \mathbb{R}$ and some $h > 0$ then

$$\|\mathscr{L}T(\lambda)\| \leq C \exp(H_M(-Re\,\lambda) + h\omega(|\lambda|)), \quad \lambda \in \mathbb{C}. \tag{7}$$

Since (a) holds we find a compact set $M \subset I\!R$ and $h > 0$ such that (6) holds. Let $m := hL_\omega$. If L is compact with $K \subset int(L)$, choose $\chi \in \mathscr{D}_{(\omega)}$ with $\operatorname{supp}\chi \subset L$ and $\chi|_K = 1$. Using (a) and Lemma 4.1 we get for all $f \in \mathscr{E}_{(\omega)}$

$$\|T(f)\| = \|T(\chi f)\| \le C\|\chi f\|_{M,h} = C\|\chi f\|_{M\cap L,h} \le C\|\chi f\|_{L,h} \le Ce^m\|\chi\|_{L,m}\|f\|_{L,m}$$

which by the remark at the beginning of the proof implies (b) for $F := \mathscr{L}T$ if we set $C_L := Ce^m\|\chi\|_{L,m}$.

If (b) holds we get T as desired as in [2], Proposition 7.3. □

Proceeding in the same way with obvious modifications we prove the following Paley-Wiener-Theorem for $\mathscr{E}'_{\{\omega\}}(X)$.

Theorem 4.3 *Let $K \subset I\!R$ be compact and X be a Banach space. Consider the following two assertions:*

(a) $T \in \mathscr{E}'_{\{\omega\}}(X)$ and $\operatorname{supp} T \subset K$.

(b) $F : I\!\!C \to X$ is entire and for all compact sets $L \subset I\!R$ with $K \subset int(L)$ and all $\epsilon > 0$ there is a $C_{L,\epsilon} > 0$ such that

$$\|F(\lambda)\| \le C_{L,\epsilon}\exp(H_L(-Re\,\lambda) + \epsilon\omega(|\lambda|)), \quad \lambda \in I\!\!C.$$

If (a) holds then (b) holds for $F := \mathscr{L}T$. If (b) holds then there exists a T with (a) and $\mathscr{L}T = F$.

We close this section with a remark concerning the behaviour of the convolution of compactly supported ultradistributions under Laplace transformation.

Remark 4.4 *If S and T are scalar valued ultradistributions with compact support then the definition of convolution yields easily $\mathscr{L}S * T(\lambda) = \mathscr{L}S(\lambda)\mathscr{L}T(\lambda)$. In the situation of Corollary 3.6 we thus have*

$$\mathscr{L}(S \otimes x *_b T \otimes y)(\lambda) = (\mathscr{L}S(\lambda)\mathscr{L}T(\lambda)b(x,y) = b(\mathscr{L}S(\lambda)x, \mathscr{L}T(\lambda)y).$$

Extension gives $\mathscr{L}(U *_b V) = b \circ (\mathscr{L}U, \mathscr{L}V)$ for vector valued ultradistributions with compact support.

5 Characterization of some solvable convolution operators

We fix Banach spaces E and D, an integer $p > 0$ and take operators $A_0, \dots, A_p \in L(D,E)$. We set

$$P := \sum_{k=0}^{p} \delta^{(k)} \otimes A_k. \tag{8}$$

Then $P \in \mathscr{D}'(L(D,E))$. We aim to characterize those P that have fundamental solutions in $\mathscr{D}'_{(\omega)}(L(E,D))$ and in $\mathscr{D}'_{\{\omega\}}(L(E,D))$ where G is called a *fundamental solution for P* if $\operatorname{supp} G \subset [0,\infty[$ and

$$P * G = \delta \otimes Id_E \qquad \text{and} \qquad G * P = \delta \otimes Id_D. \tag{9}$$

The next theorem covers the Beurling case.

Theorem 5.1 *The convolution operator P has a fundamental solution in $\mathscr{D}'_{(\omega)}(L(E,D))$ if and only if there are constants $\alpha, \beta, k > 0$ such that for all*

$$\lambda \in \Lambda := \{z \in I\!\!C : Re\,z \ge \alpha\omega(|z|) + \beta\} \tag{10}$$

the operator $\mathscr{L}P(\lambda)^{-1}$ exists and satisfies for all $\epsilon > 0$

$$\|\mathscr{L}P(\lambda)^{-1}\| \le C_\epsilon \exp(\epsilon Re\,\lambda + k\omega(|\lambda|)). \tag{11}$$

Before proving the theorem we note a lemma for convenience.

Lemma 5.2 *Let α, β be > 0 and define Λ as in* (10). *Then there exists $\tilde{\beta} > 0$ such that with $\tilde{\alpha} := \alpha K$ (K from (α)) we have*

$$\tilde{\Lambda} := \{\xi + i\eta : \xi \geq \tilde{\alpha}\omega(|\eta|) + \tilde{\beta}\} \subset \Lambda.$$

Proof. For all $\lambda = \xi + i\eta \in \mathbb{C}$ we have

$$\omega(|\lambda|) \leq \omega(|\xi| + |\eta|) \leq \omega(2\max(|\xi|, |\eta|)). \tag{12}$$

Since $\lim_{t\to\infty} \omega(t)/t = 0$ we find a $c > 0$ such that $\omega(t) \leq t/2$ for $t > c$. Now for $\lambda \notin \Lambda$ with $\xi \geq 0$ and $|\lambda| > c$ we have

$$\xi < \alpha\omega(|\lambda|) + \beta \leq |\lambda|/2 \leq \xi/2 + |\eta|/2$$

which implies $\xi \leq |\eta|$. Set $\tilde{\beta} := \max(\beta + K, c)$. Then we have for $\lambda \notin \Lambda$ with $\xi \geq 0$ that if $|\lambda| \leq c$ then $\lambda \notin \tilde{\Lambda}$ by the choice of $\tilde{\beta}$ and if $|\lambda| > c$ then by (12) and (α)

$$\xi < \alpha\omega(|\lambda|) + \beta \leq \alpha\omega(2|\eta|) + \beta \leq \alpha K\omega(|\eta|) + K + \beta \leq \tilde{\alpha}\omega(|\eta|) + \tilde{\beta},$$

and hence $\lambda \notin \tilde{\Lambda}$. □

Proof of Theorem 5.1. Let $G \in \mathscr{D}'_{(\omega)}(L(E, D))$ be a fundamental solution for P. Choose $b > a > 0$ and a function $\rho \in \mathscr{D}_{(\omega)}$ with $\rho|_{[-1,a]} = 1$ and $\rho|_{\mathbb{R}\setminus[-2,b]} = 0$. Then

$$P * (\rho G) = P * G - P * (1-\rho)G = \delta \otimes Id_E - \Phi_\rho$$
$$(\rho G) * P = G * P - (1-\rho)G * P = \delta \otimes Id_D - \Psi_\rho.$$

Applying the Laplace transform and making use of Remark 4.4 we get

$$\mathscr{L}P \circ \mathscr{L}(\rho G) = Id_E - \mathscr{L}\Phi_\rho \quad \text{and} \quad \mathscr{L}(\rho G) \circ \mathscr{L}P = Id_D - \mathscr{L}\Psi_\rho.$$

Now $\operatorname{supp}\Phi_\rho \cup \operatorname{supp}\Psi_\rho \subset [a, b]$ and by Theorem 4.2 we get constants $C_1, h > 0$ with

$$\max\left(\|\mathscr{L}\Phi_\rho(\lambda)\|, \|\mathscr{L}\Psi_\rho(\lambda)\|\right) \leq C_1 \exp(-a' Re\,\lambda + h\omega(|\lambda|)) \tag{13}$$

for all $\lambda \in \mathbb{C}$ with $Re\,\lambda \geq 0$ where $a' \in]0, a[$. Let $\alpha := h/a'$, choose $\beta > 0$ with $\beta \geq 1/a' \cdot \log(2C_1)$, and define Λ as in (10). For $\lambda \in \Lambda$ the right hand side of (13) is not greater than $1/2$. Hence the operator

$$\mathscr{L}P(\lambda)^{-1} = \mathscr{L}(\rho G)(\lambda)(Id_E - \mathscr{L}\Phi_\rho(\lambda))^{-1} = (Id_D - \mathscr{L}\Psi_\rho(\lambda))^{-1}\mathscr{L}P(\rho G)(\lambda)$$

exists and satisfies

$$\|\mathscr{L}P(\lambda)^{-1}\| \leq 2\|\mathscr{L}(\rho G)(\lambda)\|. \tag{14}$$

Applying Theorem 4.2 once more we get (11), and thus the "only if"-part is proved.

To prove the "if"-part of the theorem we can assume by Lemma 1.2 that ω is continuously differentiable. By Lemma 5.2 we may without loss of generality assume that (10) is replaced by $\Lambda := \{\xi + i\eta : \xi \geq \alpha\omega(|\eta|) + \beta\}$. Let Γ denote the boundary of Λ parametrized by

$$\gamma(\eta) := \alpha\omega(|\eta|) + \beta + i\eta, \ \eta \in \mathbb{R}.$$

Notice that γ' is bounded since ω' is bounded.

We now want to define

$$G(\phi) := \frac{1}{2\pi i}\int_\Gamma \mathscr{L}P(\lambda)^{-1}\int_{\mathbb{R}} \phi(t)e^{\lambda t}\,dt\,d\lambda. \tag{15}$$

for all $\phi \in \mathscr{D}_{(\omega)}$. We show that this operator is well defined. The inner integral equals $\hat{\phi}(i\lambda)$, and thus, if $\operatorname{supp}\phi$ is a subset of the compact $K \subset I\!R$, its absolute value can by [2], Lemma 3.3, be majorized by

$$m(K)D_b\|\phi\|_b \exp(H_K(Re\,\lambda) - b/L \cdot \omega(|\lambda|)), \; b > 0, \tag{16}$$

where m denotes Lebesgue measure and $L > 0$ is an absolut constant. If $\sup K \le c$ using (11) we can estimate the integrand in (15) from above by

$$C_\epsilon m(K)D_b\|\phi\|_b \exp((c+\epsilon)\beta)\exp(((c+\epsilon)\alpha + k - b/L)\omega(|\lambda|)) \; \epsilon, b > 0. \tag{17}$$

Fix $\epsilon > 0$ and choose $b > 0$ such that $(c+\epsilon)\alpha + k - b/L < 0$. Then (17) is by (γ) majorized by $\text{const} \cdot |\eta|^{-2}\|\phi\|_b$ for $|\eta|$ large, which is integrable. Hence (15) defines an element of $\mathscr{D}'_{(\omega)}(L(E,D))$.

We now show that $\operatorname{supp} G \subset [0,\infty[$ and proceed as in the proof of Theorem 1.6 in [3]. Notice first that the integral $\int_\Gamma$ in (15) can be replaced by $\int_{\Gamma+l}$ for arbitrary $l \in I\!N$.

We fix $\delta > 0$ and take $\phi \in \mathscr{D}_{(\omega)}$ with $\operatorname{supp}\phi \subset]-\infty, -\delta]$. We set $\epsilon := \delta/2$ and $b := kL$. Then we can majorize the integrand in (15) (with Γ replaced by $\Gamma + l$ by

$$C_\epsilon m(K)D_b\|\phi\|_b \exp(-\delta(\beta+l)/2)\exp(-\delta\alpha/2 \cdot \omega(|\eta|)).$$

Proceeding as above we thus get $\|G(\phi)\| \le \text{const}\, \exp(-l\delta/2)$ for all $l \in I\!N$ which implies $G(\phi) = 0$.

Using (8) we obtain (9) just like in the proof of Theorem 1.6 in [3]. □

Remark 5.3 This theorem extends the result in [5] from subadditive to general weight functions.

Theorem 5.4 *The convolution operator P has a fundamental solution in $\mathscr{D}'_{\{\omega\}}(L(E,D))$ if and only if, for every $\epsilon > 0$, there is a $\beta(\epsilon) > 0$ such that for all*

$$\lambda \in \Lambda_\epsilon := \{z \in \mathbb{C} : Re\, z \ge \epsilon\omega(|z|) + \beta(\epsilon)\} \tag{18}$$

the operator $\mathscr{L}P(\lambda)^{-1}$ exists and satisfies for all $\delta_1, \delta_2 > 0$

$$\|\mathscr{L}P(\lambda)^{-1}\| \le C_{\delta_1,\delta_2} \exp(\delta_1 Re\,\lambda + \delta_2\omega(|\lambda|)) \tag{19}$$

where C_{δ_1,δ_2} does not depend on ϵ.

Proof. We proceed as in the proof of Theorem 5.1. First let $G \in \mathscr{D}'_{\{\omega\}}(L(E,D))$ be a fundamental solution for P. Choose $b > a > 0$ and a ρ as before. Let $\epsilon > 0$ be arbitrary. Take $a' \in]0,a[$ and let $h := a'\epsilon$. Then by Theorem 4.3 we get a constant $C_1(\epsilon)$ such that (13) holds for all $\lambda \in \mathbb{C}$ with $Re\,\lambda \ge 0$. Choosing a positive $\beta(\epsilon) \ge 1/a' \cdot \log(2C_1(\epsilon))$ and defining Λ_ϵ as in (18) we get as before that $\mathscr{L}P(\lambda)^{-1}$ exists for all $\lambda \in \Lambda$ and satisfies (14). Another application of Theorem 4.3 finishes the proof of the "only if"-part.

To prove the "if"-part we proceed essentially as above but we have to pay a little more attention to the choice of the constants. So take a compact set $K \subset I\!R$, a $\phi \in \mathscr{D}_{\{\omega\}}(K)$, and a $c \ge \sup K$. We find $b > 0$ with $\|\phi\|_b < \infty$. By [2], Lemma 3.3, we have for all $\lambda \in \mathbb{C}$

$$|\hat{\phi}(i\lambda)| \le m(K)D_b\|\phi\|_b \exp(H_K(Re\,\lambda) - b/L\omega(|\lambda|)).$$

For all $\lambda \in \Gamma_\epsilon := \partial\Lambda_\epsilon$ we then have by (19) for all $\delta_1, \delta_2 > 0$

$$\|\mathscr{L}P(\lambda)^{-1}\hat{\phi}(i\lambda)\| \le C_{\delta_1,\delta_2} m(K)D_b\|\phi\|_b \exp((c+\delta_1)\beta(\epsilon) + ((c+\delta_1)\epsilon + \delta_2 - b/L)\omega(|\lambda|)). \tag{20}$$

Thus, if $c\epsilon < b/L$, then by suitably choosing $\delta_1, \delta_2 > 0$ the integral

$$\frac{1}{2\pi i}\int_{\Gamma_\epsilon} \mathscr{L}P(\lambda)^{-1}\hat{\phi}(i\lambda)\; d\lambda \tag{21}$$

exists and its norm is majorized by const $\cdot \|\phi\|_b$. Cauchy's integral theorem and estimations as above show that (21) does not depend on ϵ as long as $c\epsilon < b/L$. We thus have defined an element G of $\mathscr{D}'_{\{\omega\}}(L(E,D))$.

We now show that $\operatorname{supp} G \subset [0,\infty[$, so let with the notations as above be $c < 0$. Fix $\epsilon = 1$. Then the integral $\int_{\Gamma_1}$ in (21) can be replaced by $\int_{\Gamma_1 + l}$ for all $l \in I\!N$. Choosing $\delta_1 := -c/2$ and $\delta_2 < b/L$ (21) can thus in norm be estimated by const $\cdot \exp(-\delta_1(\beta(1)+l))$ for any $l \in I\!N$ which implies $G(\phi) = 0$.

Again, using (8) we obtain (9) just like in the proof of Theorem 1.6 in [3]. □

Remark 5.5 If ω is a *strong weight function*, i. e. we have $\int_1^\infty t^{-2}\omega(yt)\, dt \leq C\omega(y) + C$ for all $y \geq 1$ (see [1]), then Theorem 5.1 holds with $\epsilon = 0$ and Theorem 5.4 holds with $\delta_1 = 0$. This is due to the fact that, in this case, the Paley-Wiener-Theorems for $K = [a,b]$ hold for $L = [a,b]$, and not just for $L = [a-\epsilon, b+\epsilon]$, $\epsilon > 0$ (see section 4 and [1]).

6 Applications

We want to apply Theorems 5.1 and 5.4 to some examples and will first set some notation. Let $\varphi : [0,\infty[\to [0,\infty[$ be a C^1-function with strictly increasing derivative satisfying $\varphi(0) = \varphi'(0) = 0$ and $\varphi'(\infty) = \infty$. Assume that φ satisfies the following conditions

$$\int_0^\infty e^{-r}\varphi(r)\, dr < \infty \quad \text{and} \quad \varphi(t+1) \leq K(\varphi(t)+1).$$

Then $\omega(r) := \varphi(\log r)$ for $r \geq 1$, $\omega(r) := 0$ for $r \in [0,1]$, defines a weight function. Let $\psi := \varphi^*$ (recall that this implies $\psi^* = \varphi$ and $\varphi' \circ \psi' = \psi' \circ \varphi' = Id_{[0,\infty[}$). Then it can be shown by integration by parts and substitution that

$$\int_0^\infty \exp(-\psi'(s))\, ds < \infty. \tag{22}$$

For the following examples we take the Banach space

$$E := \{f \in C^\infty[0,1] : \|f\| := \|f\|_{1,[0,1]} = \sup_n \|f^{(n)}\|_\infty \exp(-\psi(n)) < \infty\}.$$

and $P := \delta' \otimes I - \delta \otimes A$ where A is the operator $-d/dx$ with different domains, i.e. with different boundary conditions. In any case the Banach space D is $D(A)$ supplied with the graph norm and $I : D \to E$ denotes the inclusion. Observe that $\mathscr{L}P(\lambda) = \lambda - A$ here, and hence the set where $\mathscr{L}P(\cdot)^{-1}$ exists is precisely the resolvent set $\rho(A)$ of A.

Example 6.1 Take $D(A) := \{f \in E : f' \in E, f(0) = 0\}$. Then we have that all λ with $Re\,\lambda > 0$ belong to $\rho(A)$ and $R(\lambda, A) := (\lambda - A)^{-1}$ is given by

$$R(\lambda, A)g(x) = e^{-\lambda x}\int_0^x e^{\lambda s} g(s)\, ds.$$

We abbreviate the right hand side by $f(x)$. Clearly $\|f\|_\infty \leq \|g\|$. Since we have

$$f^{(n)} = (-\lambda)^n f + \sum_{j=0}^{n-1} (-\lambda)^{n-1-j} g^{(j)} \tag{23}$$

for all $n \in I\!N$ we estimate

$$\|f^{(n)}\|_\infty \leq \left(|\lambda|^n + \sum_{j=0}^{n-1} |\lambda|^{n-1-j} e^{\psi(j)} \right) \|g\|$$

which leads to

$$\|f\| \leq \sup_n \left(|\lambda|^n e^{-\psi(n)} + \sum_{j=0}^{n-1} |\lambda|^j e^{-\psi(j+1)} \right) \|g\|$$

where we used $\psi(j) - \psi(n) \leq -\psi(n-j)$ and changed the summation index. Now $\sup_n |\lambda|^n \exp(-\psi(n)) \leq \omega(|\lambda|)$ and $\psi(j+1) - \psi(j) \geq \psi'(j)$ which implies

$$\sum_{j=0}^{n-1} |\lambda|^j e^{-\psi(j+1)} \leq e^{\omega(|\lambda|)} \left(\sum_{j=0}^{n-1} e^{-\psi'(j)} \right).$$

By (22) the series $\sum_j \exp(-\psi'(j))$ converges and hence

$$\|R(\lambda, A)g\| \leq e^{\omega(|\lambda|)}(1 + \sum_{j=0}^{\infty} e^{-\psi'(j)})\|g\|$$

for all $Re\,\lambda \geq 0$. By Theorem 5.4 P has a fundamental solution in $\mathscr{D}'_{\{\omega\},+}(L(E,D))$ (use the resolvent equation to get an estimate for $\|R(\lambda, A)\|_{L(E,D)}$). Observe that this operator is not stationary dense (for this notion see [7]).

Example 6.2 Take $D(A) := \{f \in E : f' \in E, f(0) = f'(1)\}$. Then all λ with $1 + \lambda \exp(-\lambda) \neq 0$ belong to $\rho(A)$ and

$$R(\lambda, A)g(x) = e^{-\lambda x}(c(\lambda, g) + \int_0^x e^{\lambda s} g(s)\, ds) \tag{24}$$

where

$$c(\lambda, g) = \frac{1}{1 + \lambda e^{-\lambda}} \left(g(1) - \lambda e^{-\lambda} \int_0^1 e^{\lambda s} g(s)\, ds \right).$$

Thus $\{Re\,\lambda \geq \max(0, \log|\lambda| + \log 2)\} \subset \rho(A)$. For λ in this set we have

$$|c(\lambda, g)| \leq 2(\|g\| + |\lambda| \|g\|).$$

We can estimate the resolvent as above, and again P has a fundamental solution in $\mathscr{D}'_{\{\omega\},+}(L(E,D))$. Observe that there are infinitely many λ with $1 + \lambda \exp(-\lambda) = 0$, all satisfying $|\lambda| = \exp(Re\,\lambda)$, as can be seen by factorizing entire functions.

Example 6.3 Fix a sequence (a_n) of positive real numbers such that $(a_n \exp(\psi(n))$ is summable and take $D(A) := \{f \in E : f' \in E, f(0) = \sum_{n=0}^{\infty} a_n f^{(n)}(1)\}$ (observe that the series converges). Again the resolvent of A is given by (24) where now

$$c(\lambda, g) = \frac{e^{-\lambda} \int_0^1 e^{\lambda s} g(s)\, ds \sum_{n=0}^{\infty} a_n(-\lambda)^n + \sum_{n=1}^{\infty} a_n \sum_{j=0}^{n-1} (-\lambda)^{n-1-j} g^{(j)}(1)}{1 - e^{-\lambda} \sum_{n=0}^{\infty} a_n(-\lambda)^n}.$$

Sufficient for the boundedness of the resolvent is

$$2|\sum_{n=0}^{\infty} a_n(-\lambda)^n| \leq e^{Re\,\lambda}.$$

For those λ we have

$$|c(\lambda, g)| \leq 2\|g\| \left(|\sum_{n=0}^{\infty} a_n(-\lambda)^n| + |\sum_{n=1}^{\infty} a_n \sum_{j=0}^{n-1} (-\lambda)^{n-1-j} e^{\psi(j)}| \right).$$

Now

$$|\sum_{n=0}^{\infty} a_n(-\lambda)^n| \leq \sum_{n=0}^{\infty} |a_n| e^{\psi(n)} \sup_m |\lambda|^m e^{-\psi(m)} \leq \|(a_n e^{\psi(n)})\|_1\, e^{\omega(|\lambda|)}, \tag{25}$$

and using $\psi(j)-\psi(n)\leq -\psi(n-j)\leq -\psi(n-1-j)-\psi'(n-1-j)$ for $0\leq j\leq n-1$ and changing the summation index we get

$$\begin{aligned}|\sum_{n=1}^{\infty}a_n\sum_{j=0}^{n-1}(-\lambda)^{n-1-j}e^{\psi(j)}| &\leq \sum_{n=1}^{\infty}|a_n|e^{\psi(n)}\sum_{j=0}^{n-1}|\lambda|^j e^{-\psi(j)-\psi'(j)}\\ &\leq \|(a_n e^{\psi(n)})\|_1\left(\sum_{j=0}^{\infty}e^{-\psi'(j)}\right)e^{\omega(|\lambda|)}.\end{aligned}$$

Hence we have

$$|c(\lambda,g)|\leq 2\|g\|e^{\omega(|\lambda|)}\|(a_n e^{\psi(n)})\|_1(1+\sum_{j=0}^{\infty}e^{-\psi'(j)}).$$

for λ as above. Choosing $a_n:=\exp(-\psi(n+1))$ and using (25) we see that P has a fundamental solution in $\mathscr{D}'_{(\omega),+}(L(E,D))$ but not in $\mathscr{D}'_{\{\omega\},+}(L(E,D))$.

Example 6.4 In any of the situations above also $\delta''\otimes I-\delta\otimes A^2$ and $\delta''\otimes I-\delta'\otimes(I+A)+\delta\otimes A$ have ultradistributional fundamental solutions where we now take $D:=D(A^2)$ with the graph norm in the first case.

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Universität Karlsruhe, Mathematisches Institut I
Englerstr. 2, D – 76128 Karlsruhe, Germany
e-mail: peer.kunstmann@math.uni-karlsruhe.de