Geometric Variational Problems

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Contents

Chapter 1

Introduction

These are the notes of a lecture given by the author in the summer term 2007 at the FU Berlin. The goal of the lecture was to present a detailed proof of the regularity result for two-dimensional geometric variational problems which was obtained by Tristan Rivière [27]. In doing so we had to introduce the so called Hardy- and Lorentz-spaces and study their properties. Moreover we gave a proof of the gauge transformation result of Karen Uhlenbeck.

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Chapter 2

Geometric variational problems

In this lecture we are mostly concerned with regularity properties of critical points of two-dimensional geometric variational problems. Before giving the general definition of these variational problems let us consider two very important examples. As it will turn out later, these examples already contain all of the difficulties which one encounters when studying regularity properties of these variational problems.

2.1 Surfaces of constant mean curvature in $\mathbb{R}^3$

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and for $u \in C^2(\Omega, \mathbb{R}^3)$ we define the functional

$$E_H(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dz + \frac{2H}{3}\int_\Omega u u_x \wedge u_y dz, \tag{2.1}$$

where $|\nabla u|^2 = |u_x|^2 + |u_y|^2$, $H \in \mathbb{R}$ is some constant, $dz = dxdy$ and for $a, b \in \mathbb{R}^3$ we let $a \wedge b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$. The term

$$V(u) = \frac{1}{3}\int_\Omega u u_x \wedge u_y dz$$

may be interpreted as the algebraic "volume" of the surface parametrized by u. In fact $V(u)$ measures the algebraic volume enclosed in the cone consisting of all lines joining the origin with u. (This is clear for u of the form $u(x, y) = ax+by+c$, $a, b, c \in \mathbb{R}^3$ and for smooth surfaces this follows from approximations by polyhedral surfaces.) Next we calculate the Euler-Lagrange equations of the functional E_H.

Lemma 2.1.1. *Let $u \in C^2(\Omega, \mathbb{R}^3)$ be a critical point of the functional E_H, then we have for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$*

$$0 = \int_\Omega \nabla u \nabla \varphi dz + 2H \int_\Omega \varphi u_x \wedge u_y dz. \tag{2.2}$$

Proof. Since u is a critical point of E_H we know that for all $\varphi \in C_c^\infty(\Omega, \mathbb{R}^3)$

$$\frac{d}{dt} E_H(u + t\varphi)|_{t=0} = 0. \tag{2.3}$$

Now

$$E_H(u+t\varphi)=\frac{1}{2}\int_\Omega |\nabla u|^2dz+t\int_\Omega \nabla u\nabla\varphi dz+\frac{t^2}{2}\int_\Omega|\nabla\varphi|^2+V(u+t\varphi) \tag{2.4}$$

and for the volume term we calculate

$$\begin{aligned}3V(u+t\varphi)=&\int_\Omega (u+t\varphi)(u+t\varphi)_x\wedge(u+t\varphi)_y dz\\ =&3V(u)+t\int_\Omega(\varphi u_x\wedge u_y+u\varphi_x\wedge u_y+uu_x\wedge\varphi_y)dz\\ &+2t^2\int_\Omega(u\varphi_x\wedge\varphi_y+\varphi(u_x\wedge\varphi_y+\varphi_x\wedge u_y))dz+3t^3V(\varphi).\end{aligned} \tag{2.5}$$

The last two formulas guarantee the existence of the derivative in (2.3) and we get

$$\begin{aligned}\frac{d}{dt}E_H(u+t\varphi)|_{t=0}=&\int_\Omega\nabla u\nabla\varphi dz+\frac{2H}{3}\int_\Omega(\varphi u_x\wedge u_y+u\varphi_x\wedge u_y+uu_x\wedge\varphi_y)dz\\ =&\int_\Omega\nabla u\nabla\varphi dz+\frac{2H}{3}\int_\Omega(\varphi u_x\wedge u_y-\varphi_x u\wedge u_y-\varphi_y u_x\wedge u)dz\\ =&\int_\Omega\nabla u\nabla\varphi dz+2H\int_\Omega\varphi u_x\wedge u_y dz+\frac{2H}{3}\int_\Omega\varphi(u\wedge u_{xy}+u_{xy}\wedge u)dz\\ =&\int_\Omega\nabla u\nabla\varphi dz+2H\int_\Omega\varphi u_x\wedge u_y dz,\end{aligned} \tag{2.6}$$

where we used integration by parts and the facts that $a\wedge b=-b\wedge a$ and $a(b\wedge c)=-b(a\wedge c)=-c(b\wedge a)$. Altogether this proves the Lemma. □

We see that critical points of the functional E_H are solutions of the partial differential equation

$$\Delta u=2Hu_x\wedge u_y \quad \text{in} \quad \Omega. \tag{2.7}$$

Up to now this equation was derived under the assumption that $u\in C^2(\Omega,\mathbb{R}^3)$. But from the definition of the functional E_H we see that it is already well-defined if $u\in W^{1,2}\cap L^\infty(\Omega,\mathbb{R}^3)$ (here $W^{1,2}(\Omega,\mathbb{R}^3)$ is the Sobolev-space of maps from Ω into $\mathbb{R}^3$ whose first weak derivative exists and is square integrable). Moreover the weak formulation of equation (2.7) (see (2.2)) is well-defined if $u\in W^{1,2}(\Omega,\mathbb{R}^3)$.

As a remark we like to mention that critical points of the functional E_H do not necessarily represent surfaces of constant mean curvature in $\mathbb{R}^3$. This is only true if the surfaces are additionally confomal, i.e. they satisfy $|u_x|^2-|u_y|^2=0=u_xu_y$. For more on surfaces of constant mean curvature, including existence results, see for example [35].

2.2 Harmonic maps

Let $u\in C^2(\Omega,\mathbb{R}^d)$ and let $g_{ij}\in C^2(\mathbb{R}^d)$ for all $i,j\in\{1,\dots,d\}$. Moreover we assume that $g=(g_{ij})$ is positive definit and symmetric. Then we define the Dirichlet energy

$$E(u)=\frac{1}{2}\int_\Omega g_{ij}(u)\nabla u^i\nabla u^j dz. \tag{2.8}$$

As above we calculate the Euler-Lagrange equations of E in the following Lemma.

Lemma 2.2.1. *Let $u \in C^2(\Omega, \mathbb{R}^d)$ be a critical point of E, then we have for all $1 \le i \le d$*

$$\Delta u^i + \Gamma^i_{jk} \nabla u^j \nabla u^k = 0, \tag{2.9}$$

where we used the Einstein summation convention and where $\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{kj,m})$, $(g^{im}) = (g_{im})^{-1}$ and $g_{jm,k} = \frac{\partial}{\partial x_k} g_{jm}$.

Proof. As in the proof of Lemma 2.1.1 we calculate for all $\varphi \in C^\infty_c(\Omega, \mathbb{R}^d)$

$$\begin{aligned} 0 =& \frac{d}{dt} E(u + t\varphi)|_{t=0} \\ =& \int_\Omega (g_{ij}(u) \nabla u^i \nabla \varphi^j + \frac{1}{2} g_{ij,k}(u) \nabla u^i \nabla u^j \varphi^k) dz \\ =& - \int_\Omega \varphi^j (g_{ij}(u) \Delta u^i + g_{ij,k} \nabla u^i \nabla u^k - \frac{1}{2} g_{ik,j} \nabla u^i \nabla u^k) dz. \end{aligned} \tag{2.10}$$

From the definition of Γ and the fact that the last identity is true for all $\varphi \in C^\infty_c(\Omega, \mathbb{R}^d)$ we easily conlude (2.9). □

Again we remark that the Dirichlet energy is well defined if $u \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^d)$. Moreover there exists a well-defined weak version of the equation (2.9) in this situation.
An interesting special case of the Dirichlet energy is given if we have a smooth and compact d-dimensional Riemannian manifold (N, g) without boundary and a map $u \in C^1(\Omega, \mathbb{R}^d)$. In this situation the Dirichlet defined above agrees with the Dirichlet energy for the map u in local coordinates.

2.3 Two-dimensional geometric variational problems

In this section we consider critical points of more general functionals. More precisely we study functionals of the form

$$E(u, \Omega) = \frac{1}{2} \int_\Omega e(u(x), \nabla u(x)) dx, \tag{2.11}$$

where Ω is a domain in $\mathbb{R}^2$ (wlog containing the origin) and $e : \mathbb{R}^d \times \mathbb{R}^{2d} \to \mathbb{R}$ is C^1 with respect to the first d variables and C^2 with respect to the remaining $2d$ variables. Additionally we assume that there exist $0 < \lambda \le \Lambda < \infty$ such that

$$\lambda |p|^2 \le e(u, p) \le \Lambda |p|^2, \tag{2.12}$$

for all $(u, p) \in \mathbb{R}^d \times \mathbb{R}^{2d}$. Moreover we assume that E is invariant under homotheties and rotations, i.e.

$$E(u \circ \phi, \Omega‘) = E(u, \Omega) \tag{2.13}$$

for all $u \in C^1(\Omega, \mathbb{R}^d)$ if $\phi(x) = rx$, $r \ne 0$, and $\Omega‘ = \frac{1}{r}\Omega$ and

$$E(u \circ \Psi, \Omega) = E(u, \Omega) \tag{2.14}$$

for all $u \in C^1(\Omega, \mathbb{R}^d)$ if $\psi(x) = S(\theta)x$ where $S(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.
In the next Theorem, which is originally due to M. Grüter [16], we classify all functionals which satisfy the above conditions.

Theorem 2.3.1. *Let $E(u,\Omega)$ be as above. Then there exists a positive definit symmetric matrix $g=(g_{ij})_{i,j=1,\dots,d}$ and a skew-symmetric matrix $b=(b_{ij})_{i,j=1,\dots,d}$ such that*

$$E(u,\Omega)=\frac{1}{2}\int_\Omega (g_{ij}(u)\nabla u^i\nabla u^j+b_{ij}(u)\det(\nabla u^i,\nabla u^j))dx. \tag{2.15}$$

Proof. The proof is taken from [20]. Using (2.13) and the transformation formula we calculate

$$\begin{aligned}\int_{\Omega'} e(u(ry),\nabla u(ry)r)dy &= \int_\Omega e(u(x),\nabla u(x))dx\\ &= \int_{\Omega'} e(u(ry),\nabla u(ry))r^2 dy.\end{aligned} \tag{2.16}$$

Since this is true for all $u\in C^1(\Omega,\mathbb{R}^d)$ we conclude that $\forall (x,u,p)\in\Omega\times\mathbb{R}^d\times\mathbb{R}^{2d}$ we have

$$e(u,rp)=r^2e(u,p). \tag{2.17}$$

Now we differentiate the identity (2.17) twice with respect to r and let $r\to 0$ to get

$$e(u,p)=\frac{1}{2}e_{p^i_\alpha p^j_\beta}(u,0)p^i_\alpha p^j_\beta. \tag{2.18}$$

Next we define $A^{ij}_{\alpha\beta}(u)=e_{p^i_\alpha p^j_\beta}(u,0)$ and $A^{ij}=(A^{ij}_{\alpha\beta})_{\alpha,\beta=1,2}$. Due to the symmetry of the second derivatives we have

$$A^{ij}_{\alpha\beta}=A^{ji}_{\beta\alpha},\quad \text{i.e.}\quad A^{ji}=(A^{ij})^T. \tag{2.19}$$

Because of (2.18) we know that

$$E(u,\Omega)=\frac{1}{2}\int_\Omega (\nabla u^i)^T A^{ij}(u)\nabla u^j dx. \tag{2.20}$$

Using (2.14) and arguing as above we see that

$$e(u,S(\theta)p)=e(u,p). \tag{2.21}$$

Combining this with (2.20) we get

$$(p^i)^T A_{ij}p^j=(p^i)^T(S(\theta)^TA^{ij}S(\theta))p^j. \tag{2.22}$$

Since this is true for all p we get

$$A^{ij}=S(\theta)^TA^{ij}S(\theta). \tag{2.23}$$

Choosing $\theta=\frac{\pi}{2}$ we calculate for all $i,j\in\{1,\dots,d\}$

$$\begin{aligned}A^{ij}_{11}&=A^{ij}_{22}\\ A^{ij}_{12}&=-A^{ij}_{21}.\end{aligned} \tag{2.24}$$

Defining $g_{ij}=A^{ij}_{11}$ and $b_{ij}=A^{ij}_{12}$ the claim follows from (2.19) and (2.20). □

Now it is easy to see why surfaces of constant mean curvature and harmonic maps are the model problems of two-dimensional geometric variational problems. Namely, in the case of harmonic maps we have $b = (b_{ij}) = 0$ and in the case of surfaces of constant mean curvature in $\mathbb{R}^3$ we have $g_{ij} = \delta_{ij}$ and $b = \frac{4H}{3}\begin{pmatrix} 0 & u^3 & -u^2 \\ -u^3 & 0 & u^1 \\ u^2 & -u^1 & 0 \end{pmatrix}$.
Next we also calculate the Euler-Lagrange equations of critical points of the functional given in (2.15).

Lemma 2.3.2. *Let $u \in C^2(\Omega, \mathbb{R}^d)$ be a critical point of the functional E given by* (2.15)*, then we have for all $1 \le i \le d$*

$$\Delta u^i + \Gamma^i_{jk} \nabla u^j \nabla u^k = g^{im}(b_{mj,k} + b_{jk,m} + b_{km,j}) \det(\nabla u^j, \nabla u^k). \tag{2.25}$$

Proof. Since we already calculated

$$\frac{d}{dt}|_{t=0} \frac{1}{2} \int_\Omega g_{ij}(u + t\varphi) \nabla (u + t\varphi)^i \nabla (u + t\varphi)^j dx$$

in the proof of Lemma 2.2.1 it remains to calculate the derivative of the second part of the functional E and this is left as an exercise. □

As in the two special cases considered before we see that two-dimensional conformally invariant variational problems satisfying the conditions from the beginning of this section are well defined in $W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^d)$. Moreover one can define weak solutions of the Euler-Lagrange equations (2.25) in this space.
One question which arises naturally in this situation is:

Are weak solutions $u \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^d)$ of (2.25) smooth?

One might try to attack this question by using standard L^p-estimates. In the next theorem we recall the L^p-estimates (see e.g. [14], [13], [38]).

Theorem 2.3.3. *Let $1 < p < \infty$, $f \in L^p(\Omega, \mathbb{R}^d)$ and let $u \in W^{2,p}_{loc} \cap L^p(\Omega, \mathbb{R}^d)$ be a solution of*

$$\Delta u = f \quad in \quad \Omega. \tag{2.26}$$

Then we have for all $\Omega' \subset\subset \Omega$

$$||u||_{W^{2,p}(\Omega',\mathbb{R}^d)} \le c(||u||_{L^p(\Omega,\mathbb{R}^d)} + ||f||_{L^p(\Omega,\mathbb{R}^d)}). \tag{2.27}$$

From the explicit form of (2.25) we see that solutions $u \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^d)$ solve an equation of the form (2.26) but in our situation we only have $f \in L^1(\Omega, \mathbb{R}^d)$ (at least if we look at the problem in a "rough" way). For this situation there is an easy counterexample to the estimate (2.27) in two dimensions given by $f(x) = \frac{1}{|x|^2(\ln(\frac{1}{|x|}))^2}$. It is easy to see that

$$f \in L^1(B_{\frac{1}{2}})$$
$$f \notin L^p(B_{\frac{1}{2}}),$$

for any $p > 1$. Since f is radial we can find a solution u of the equation

$$\Delta u = f$$

by introducing polar coordinates and solving

$$u``(r) + \frac{1}{r}u`(r) = \frac{1}{r^2(\ln(\frac{1}{r}))^2}.$$

Equivalently we have

$$(ru`(r))` = \frac{1}{r(\ln(\frac{1}{r}))^2},$$

which can be integrated to yield

$$u`(r) = \frac{1}{r\ln(\frac{1}{r})}.$$

Differentiating this again gives

$$u``(r) = -\frac{1}{r^2\ln(\frac{1}{r})} + \frac{1}{r^2(\ln(\frac{1}{r}))^2}.$$

The second term in this expression is in $L^1(B_{\frac{1}{2}})$ but the first term is not in this space and therefore $\nabla^2 u \notin L^1(B_{\frac{1}{2}})$ for this solution.

This shows that the classical estimates for partial differential equations are not directly applicable to prove the regularity of weak solutions of (2.25).

Chapter 3

Harmonic maps into spheres

In section 1.2 we already defined harmonic maps from a domain in $\mathbb{R}^2$ into a Riemannian manifold N. The definition we gave in this section has one major disadvantage, namely that it already requires the harmonic map to be at least continuous. This is used to guarentee that the image of a small ball in Ω is maped into the domain of definiton of a chart in N so that we can write down the Euler-Lagrange equation in local coordinates. In the last section of this chapter we will see that once a harmonic map is continuous it is automatically smooth. Therefore assuming that the harmonic map is continuous is too restrictive. This is also supported by the fact that we only want to consider weak solutions which are in $W^{1,2} \cap L^\infty$ and this space does not embedd into C^0 in two dimensions. An example for this fact is given by the function $f : B_{\frac{1}{2}} \subset \mathbb{R}^2 \to \mathbb{R}$, $f(x) = \ln\ln(\frac{1}{|x|})$ (Exercise!). Because of these facts we give a different definition of harmonic maps into spheres. For continuous maps this definition will be equivalent to the one given in section 1.2.

3.1 Euler-Lagrange equation and a conservation law

Let $B = B_1$ be the unit ball in $\mathbb{R}^2$ and let $S^n = \{x \in \mathbb{R}^{n+1} | |x| = 1\}$ be the round sphere in $\mathbb{R}^{n+1}$. We define the Sobolev space $W^{1,2}$ for maps from B into S^n by

$$W^{1,2}(B, S^n) = \{u \in W^{1,2}(B, \mathbb{R}^{n+1}) | |u(x)| = 1 \quad \text{for a.e.} \quad x \in B\}. \tag{3.1}$$

Definition 3.1.1. *A critical point $u \in W^{1,2}(B, S^n)$ of the functional $E(u) = \frac{1}{2}\int_B |\nabla u|^2 dx$ is called a (weakly) harmonic map from B into S^n.*

In the next Lemma we calculate the Euler-Lagrange equation of weakly harmonic maps.

Lemma 3.1.2. *Let $u \in W^{1,2}(B, S^n)$ be a harmonic map from B into S^n, then we have*

$$\int_B \nabla u \nabla v dx = \int_B |\nabla u|^2 u v dx, \tag{3.2}$$

for all $v \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^{n+1})$.

Proof. Let $v \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^{n+1})$. Then we have for t small enough that $|u + tv| \neq 0$. Therefore we can define

$$w_t = \frac{u + tv}{|u + tv|}. \tag{3.3}$$

Then it is easy to check (Exercise!) that $w_t \in W^{1,2}(B,S^n)$ for every t small enough. This means we can define $E(w_t)$ and since $w_0 = u$ and u is harmonic we have

$$0 = \frac{d}{dt}|_{t=0}E(w_t) = \int_B \nabla u \nabla (\frac{d}{dt}|_{t=0}w_t)dx. \tag{3.4}$$

Now

$$\begin{aligned}\nabla \frac{d}{dt}w_t =& \frac{\nabla v}{|u+tv|} - \frac{v[(u+tv)\cdot\nabla(u+tv)]}{|u+tv|^3} - \frac{\nabla([(u+tv)\cdot v](u+tv))}{|u+tv|^3} \\ &+ \frac{[(u+tv)\cdot v](u+tv)[(u+tv)\cdot\nabla(u+tv)]}{|u+tv|^5}\end{aligned}$$

and therefore ($|u|^2 = 1$)

$$\nabla \frac{d}{dt}|_{t=0}w_t = \nabla v - \nabla[(u\cdot v)u]. \tag{3.5}$$

Inserting this into (3.4) we get

$$\begin{aligned}0 &= \int_B \nabla u(\nabla v - \nabla[(u\cdot v)u])dx \\ &= \int_B \nabla u \nabla v dx - \int_B |\nabla u|^2 u\cdot v dx,\end{aligned} \tag{3.6}$$

where we used again that $|u(x)|^2 = 1$ a.e. □

In the rest of this section we want to derive an equation which is equivalent to (3.2) but which is in divergence form. This conservation law has been discovered independently by Chen [6] and Shatah [31] (see also [19]). As we will see during the derivation of this equation it relies heavily on the special symmetry of the target manifold S^n.

Theorem 3.1.3. *$u \in W^{1,2}(B,S^n)$ is a weakly harmonic map iff*

$$\operatorname{div}(u^i\nabla u^j - u^j\nabla u^i) = 0, \tag{3.7}$$

for all $i,j \in \{1,\dots,n+1\}$.

Proof. $\Rightarrow$:
Let $u \in W^{1,2}(B,S^n)$ and let $a = (a_{ij})$ be the skew-symmetric matrix whose i,j-component is equal to 1 and the remaining components in the lower triangle are equal to 0. Next we consider $\varphi \in C_c^\infty(B,\mathbb{R})$ and we let $R(t) = e^{\varphi a t}$ (t small) be the solution of the flow generated by φa with $R(0) = id$. Since $\varphi a \in so(n+1)$ we get that $R \in SO(n+1)$. Namely

$$\begin{aligned}\frac{d}{dt}R^T(t)R(t) &= \varphi(R^T(t)aR + R^T(t)a^T R(t)) \\ &= 0,\end{aligned}$$

which also implies $R^{-1}(t) = R^T(t)$ and

$$\begin{aligned}\frac{d}{dt}\det(R(t)) &= \det(R(t))\operatorname{tr}(R^{-1}(t)\frac{d}{dt}R(t)) \\ &= \det(R(t))\operatorname{tr}(R^T(t)aR(t)) \\ &= 0.\end{aligned}$$

Now $R(0)u = u$ is harmonic and therefore

$$\frac{d}{dt}|_{t=0} E(R(t)u) = 0. \tag{3.8}$$

Next we use the expansion $R(t)u = u + t\varphi a u + o(t)$ and calculate

$$\begin{aligned} E(R(t)u) &= E(u) + t\int_B (\nabla u)^T \nabla(\varphi a u)dx + o(t) \\ &= E(u) + t\int_B (\nabla u)^T a(u\nabla\varphi + \varphi\nabla u)dx + o(t) \\ &= E(u) + t\int_B (\nabla u)^T a u \nabla\varphi dx + o(t), \end{aligned} \tag{3.9}$$

where we used that $(\nabla u)^T a\nabla u = 0$ (since $a \in so(n+1)$). Inserting (3.9) into (3.8) we get that

$$\operatorname{div}(u^i\nabla u^j - u^j\nabla u^i) = 0 \tag{3.10}$$

weakly. Since the $\frac{n(n+1)}{2}$-dimensional vector space $so(n+1)$ is spanned by the matrices a the claim follows.

<u>$\Leftarrow$:</u>
Let $\varphi \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^{n+1})$ and define

$$\psi = \varphi - (u\varphi)u. \tag{3.11}$$

Now we calculate for a.e. $x \in B$ ($|u|^2 = 1$)

$$\begin{aligned} \nabla\varphi\nabla u - (u\varphi)|\nabla u|^2 =& \nabla u\nabla\psi \\ =& |u|^2\nabla u\nabla\psi - (u\nabla\psi)(u\nabla u) \\ =& \sum_{i,j\in\{1,\dots,n+1\}} (u^iu^i\nabla u^j\nabla\psi^j - u^i\nabla\psi^iu^j\nabla u^j) \\ =& \sum_{i<j}(u^iu^i\nabla u^j\nabla\psi^j + u^ju^j\nabla u^i\nabla\psi^i - u^iu^j\nabla\psi^i\nabla u^j - u^iu^j\nabla\psi^j\nabla u^i) \\ =& \sum_{i<j}(u^i\nabla u^j - u^j\nabla u^i)(u^i\nabla\psi^j - u^j\nabla\psi^i) \\ =& \sum_{i<j}(u^i\nabla u^j - u^j\nabla u^i)\nabla(u^i\psi^j - u^j\psi^i) \\ =& \sum_{i<j}(u^i\nabla u^j - u^j\nabla u^i)\nabla(u^i\varphi^j - u^j\varphi^i), \end{aligned} \tag{3.12}$$

where we used that

$$\begin{aligned} u^i\psi^j - u^j\psi^i &= u^i\varphi^j - u^j\varphi^i - (u\varphi)(u^iu^j - u^ju^i) \\ &= u^i\varphi^j - u^j\varphi^i, \end{aligned}$$

for every $1 \leq i < j \leq n+1$ and

$$\sum_{i<j}(u^i\nabla u^j - u^j\nabla u^i)(\nabla u^i\psi^j - \nabla u^j\psi^i) = (u\nabla u)(\psi\nabla u) - (u\psi)(\nabla u\nabla u)$$
$$= 0.$$

Now we let $\alpha^{ij} = (u^i\varphi^j - u^j\varphi^i)$ for every $1 \leq i < j \leq n+1$. Then we have $\alpha^{ij} \in W_0^{1,2} \cap L^\infty(B, \mathbb{R}^{n+1})$ and therefore we can use (3.12) and our assumption to get

$$\begin{aligned} 0 &= \sum_{i<j}\int_B \nabla\alpha^{ij}(u^i\nabla u^j - u^j\nabla u^i)dx \\ &= \int_B (\nabla\varphi\nabla u - \varphi u|\nabla u|^2)dx. \end{aligned} \tag{3.13}$$

This shows that u is harmonic. □

Remark 3.1.4. *As can be seen from the above proof the existence of the conservation law (3.7) is a consequence of the invariance under rotations of the Dirichlet energy for maps into S^n. This can therefore be seen as a special case of Noether's theorem (see [19]).*

An easy consequence of the above conservation law is the so called weak compactness of weakly harmonic maps into spheres.

Corollary 3.1.5. *Let $u_k \in W^{1,2}(B, S^n)$ be a sequence of weakly harmonic maps and assume that there exists some $u \in W^{1,2}(B, S^n)$ such that $u_k \rightharpoonup u$ weakly in $W^{1,2}(B, S^n)$, then u is weakly harmonic.*

Proof. Applying Theorem 3.1.3 we have that

$$\int_B (u_k^i\nabla u_k^j - u_k^j\nabla u_k^i)\nabla\alpha dx = 0, \tag{3.14}$$

for every $\alpha \in C_c^\infty(B, \mathbb{R}^{n+1})$ and every $i, j \in \{1, \ldots, n+1\}$. Since $u_k \rightharpoonup u$ weakly in $W^{1,2}(B, S^n)$ we can apply Rellich's theorem to get

$$u_k \to u \tag{3.15}$$

strongly in $L^2(B, \mathbb{R}^{n+1})$ and therefore

$$u_k^i\nabla u_k^j - u_k^j\nabla u_k^i \rightharpoonup u^i\nabla u^j - u^j\nabla u^i \tag{3.16}$$

weakly in $L^1(B, \mathbb{R})$ for every $i, j \in \{1, \ldots, n+1\}$. This means that we can pass to the limit in (3.14) and get

$$\int_B (u^i\nabla u^j - u^j\nabla u^i)\nabla\alpha dx = 0, \tag{3.17}$$

for every $\alpha \in C_c^\infty(B, \mathbb{R}^{n+1})$ and every $i, j \in \{1, \ldots, n+1\}$. Applying once more Theorem 3.1.3 we conclude that u is weakly harmonic. □

3.2 Wente‘s inequality

In this section we prove the famous Wente-inequality, see [40], [3], [19] .

Theorem 3.2.1. *Let $a, b \in W^{1,2}(B, \mathbb{R})$ and let u be a weak solution of*

$$\begin{aligned} -\Delta u &= \{a, b\} \quad in \quad B \\ u &= 0 \quad on \quad \partial B, \end{aligned} \tag{3.18}$$

where $\{a, b\} = a_x b_y - a_y b_x$. Then $u \in C^0 \cap W^{1,2}(B, \mathbb{R})$ and moreover

$$||u||_{L^\infty(B)} + ||\nabla u||_{L^2(B)} \leq c||\nabla a||_{L^2(B)}||\nabla b||_{L^2(B)}. \tag{3.19}$$

Remark 3.2.2. *In this situation weak solution means $u \in W^{1,r}(B, \mathbb{R})$ for some $r > 1$. From "classical" theory we only get that weak solutions u of $\Delta u = f$ in B, $f \in L^1(B, \mathbb{R})$, with $u = 0$ on ∂B, are in L^q for every $q < \infty$ and in $W^{1,p}$ for every $p < 2$. Namely we can write*

$$u(x) = \frac{1}{2\pi} \int_B f(y)(\ln(|x - y|) - \ln(|\frac{x}{|x|} - |x|y)|)dy.$$

This implies

$$|\nabla u(x)| \leq c \int_B \frac{|f(y)|}{|x - y|} dy,$$

and since $||\frac{1}{|x|}||_{L^p(B)} \leq c$ for every $p < 2$ we get from standard estimates for convolutions that

$$||u||_{L^q(B)} + ||\nabla u||_{L^p(B)} \leq c||f||_{L^1(B)}, \tag{3.20}$$

for every $q < \infty$ and $p < 2$. In chapter 4 on Lorentz spaces we will see that we can even improve this estimate by replacing the L^p-norm by the weak L^2-norm of ∇u (see Theorem 5.2.4).

Remark 3.2.3. *Recent interesting results of Bourgain & Brezis [2] and Brezis & van Schaftingen [4] show that weak solutions $u : B \to \mathbb{R}^2$ of*

$$\begin{aligned} \Delta u &= f \quad in \quad B, \\ u &= 0 \quad on \quad \partial B, \end{aligned} \tag{3.21}$$

where $f \in L^1(B, \mathbb{R}^2)$ with $\operatorname{div} f \in L^1(B)$, satisfy $u \in W^{1,2} \cap C^0(B, \mathbb{R}^2)$ with

$$||u||_{L^\infty} + ||\nabla u||_{L^2} \leq c||f||_{L^1}. \tag{3.22}$$

Proof. (of Theorem 3.2.1) We begin by considering $a, b \in W^{1,2} \cap C^\infty(B, \mathbb{R})$. Since the Green‘s function of $-\Delta$ on B is given by $\frac{1}{2\pi} \ln(\frac{1}{|x|})$ we have

$$u(0) = \frac{1}{2\pi} \int_B \{a, b\}(x) \ln(\frac{1}{|x|}) dx. \tag{3.23}$$

Now we introduce polar coordinates and note that for $v(r, \theta) = a(r \cos \theta, r \sin \theta)$ we have

$$\begin{aligned} v_r &= a_x \cos \theta + a_y \sin \theta, \\ v_\theta &= -a_x r \sin \theta + a_y r \cos \theta. \end{aligned}$$

and the same for $w(r,\theta) = b(r\cos\theta, r\sin\theta)$. Therefore we calculate

$$\begin{aligned} v_r w_\theta - v_\theta w_r &= (a_x b_y - a_y b_x)(r\cos\theta, r\sin\theta)(r\cos^2\theta + r\sin^2\theta) \\ &= r\{a,b\}(r\cos\theta, r\sin\theta). \end{aligned} \tag{3.24}$$

Hence we see that

$$\begin{aligned} u(0) &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \ln(\frac{1}{r})(v_r w_\theta - v_\theta w_r) dr d\theta \\ &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \ln(\frac{1}{r})[(v w_\theta)_r - (v w_r)_\theta] dr d\theta \\ &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi} \frac{1}{r} v w_\theta dr d\theta, \end{aligned} \tag{3.25}$$

where we used that $\ln(\frac{1}{r})(vw_\theta) = r\ln(\frac{1}{r})[a(b_y\cos\theta - b_x\sin\theta)]$, which implies for the boundary terms $\ln(\frac{1}{r})(vw_\theta)|_{r=1} = 0 = \ln(\frac{1}{r})(vw_\theta)|_{r=0}$. Now by denoting $\overline{v}_r = \frac{1}{2\pi}\int_0^{2\pi} v(r,\theta)d\theta$ we can estimate by using Hölder's- and Poincare's inequality

$$\begin{aligned} |\int_0^{2\pi} v w_\theta d\theta| &= |\int_0^{2\pi} (v - \overline{v}_r) w_\theta d\theta| \\ &\leq (\int_0^{2\pi} |v - \overline{v}_r|^2 d\theta)^{\frac{1}{2}} (\int_0^{2\pi} |w_\theta|^2 d\theta)^{\frac{1}{2}} \\ &\leq c(\int_0^{2\pi} |v_\theta|^2 d\theta)^{\frac{1}{2}} (\int_0^{2\pi} |w_\theta|^2 d\theta)^{\frac{1}{2}}. \end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26) and transforming back we get

$$\begin{aligned} |u(0)| &\leq c\int_0^1 (\int_0^{2\pi} |v_\theta|^2 d\theta)^{\frac{1}{2}} (\int_0^{2\pi} |w_\theta|^2 d\theta)^{\frac{1}{2}} \frac{dr}{r} \\ &\leq c||\nabla a||_{L^2(B)} ||\nabla b||_{L^2(B)}. \end{aligned} \tag{3.27}$$

For a general point $x_0 \in B$ we consider the Möbius transformation $T : B \to B$, $T(z) = \frac{z+x_0}{1+z\overline{x_0}}$ (remember that T is holomorphic) with $T(0) = x_0$. From complex analysis we know that T is conformal and therefore we can write the differential ∇T in the form

$$\nabla T = T_0 \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}. \tag{3.28}$$

Now elementary calculations show that

$$\begin{aligned} -\Delta(u\circ T) &= -\det(\nabla T)(\Delta u)\circ T \\ &= \det(\nabla T)\{a,b\}\circ T \\ &= \{a\circ T, b\circ T\}. \end{aligned} \tag{3.29}$$

Therefore we can apply the previous estimate (3.27) to get

$$\begin{aligned} |u(x_0)| &= |u(T(0))| \\ &\leq c||\nabla(a\circ T)||_{L^2(B)} ||\nabla(b\circ T)||_{L^2(B)}. \end{aligned} \tag{3.30}$$

Another elementary calculation shows that the Dirichlet energy is invariant under Möbius transformations and therefore we finally get for every $x_0 \in B$

$$|u(x_0)| \leq c||\nabla a||_{L^2(B)}||\nabla b||_{L^2(B)}. \tag{3.31}$$

This proves the L^∞-estimate for smooth a and b. To get the estimate for the L^2-norm of the gradient of u we integrate by parts and use the boundary condition to get

$$\begin{aligned}\int_B |\nabla u|^2 dx &= \int_B u(-\Delta u) dx \\ &= \int_B u\{a,b\} dx \\ &\leq ||u||_{L^\infty(B)}||\{a,b\}||_{L^1(B)} \\ &\leq c||\nabla a||^2_{L^2(B)}||\nabla b||^2_{L^2(B)},\end{aligned} \tag{3.32}$$

where we used (3.31) and Hölder‘s inequality in the last estimate. This finishes the proof of (3.19) for smooth a and b.
In the general case we choose two sequences $a_n, b_n \in W^{1,2} \cap C^\infty(B,\mathbb{R})$ such that $a_n \to a$ and $b_n \to b$ in $W^{1,2}(B,\mathbb{R})$ and we let $u_n \in C^\infty(B,\mathbb{R})$ be the solution of (3.18) with a, b replaced by a_n and b_n. Since $\{a,b\}$ is linear in a and b and $\{a,b\} = -\{b,a\}$ we see that for every $n,m \in \mathbb{N}$

$$\begin{aligned}-\Delta(u_n - u_m) &= \{a_n, b_n\} - \{a_m, b_m\} \\ &= \{a_n - a_m, b_n\} - \{a_m, b_m - b_n\}.\end{aligned} \tag{3.33}$$

If we then apply estimate (3.19) to $u_n - u_m$ we see that u_n is a Cauchy sequence in the complete space $W^{1,2} \cap L^\infty(B,\mathbb{R})$ and therefore there exists $U \in W^{1,2} \cap L^\infty(B,\mathbb{R})$ such that $u_n \to U$ in $W^{1,2} \cap L^\infty(B,\mathbb{R})$ and

$$||U||_{L^\infty(B)} + ||\nabla U||_{L^2(B)} \leq c||\nabla a||_{L^2(B)}||\nabla b||_{L^2(B)}. \tag{3.34}$$

Since $u_n \in C^\infty(B,\mathbb{R})$ and $||u_n - U||_{L^\infty(B)} \to 0$ we know additionally that $U \in C^0(B,\mathbb{R})$. Moreover, from (3.20), we get that $u_n \to u$ strongly in $W^{1,p}(B,\mathbb{R})$ for every $p < 2$. Since $W^{1,2} \cap L^\infty(B,\mathbb{R}) \subset W^{1,p}(B,\mathbb{R})$ this implies that $U = u$ and finishes the proof of Wente‘s inequality. □

Remark 3.2.4. *From the proof of the Wente inequality we easily see that the statement remains true if we replace the Dirichlet boundary condition by the Neumann boundary condition* $\frac{\partial u}{\partial \nu} = 0$ *on* ∂B *and if we assume that* $\int_B u(x)dx = 0$. *The reason for this is that the Green‘s function for the Neumann problem on* $B \subset \mathbb{R}^2$ *is also of the form* $K(x,y) = c\log|x-y|$.

3.3 Continuity of sphere-valued harmonic maps

The goal of this section is to prove the continuity of weakly harmonic maps into spheres. This result was first proved by F. Hélein [17]. First we need the following Hodge decomposition for one forms in B.

Lemma 3.3.1. *Let $1 < p < \infty$ and let $\omega \in L^p(B, \Lambda^1\mathbb{R}^2)$, $\omega = \omega_1 dx^1 + \omega_2 dx^2$, be a one-form. Then there exist two functions $\alpha, \beta \in W^{1,p}(B)$ such that $\omega = \nabla\alpha + \nabla^\perp\beta$ ($\star d = \nabla^\perp = (\partial_y, -\partial_x)$ and $||\alpha||_{W^{1,p}(B)} + ||\beta||_{W^{1,p}(B)} \leq c||\omega||_{L^p(B)}$. If we assume additionally that $\sum_{i=1}^2 \nabla_i\omega_i = 0$ (i.e. $\delta\omega = 0$) weakly, then we have $\omega = \star\beta$ (i.e. $\omega = \nabla^\perp\beta$) and $||\beta||_{W^{1,p}(B)} \leq c||\omega||_{L^p(B)}$. If we know in this case that $\omega \cdot \nu = 0$ on ∂B then we can choose β such that $\beta = 0$ on ∂B.*

Proof. First of all we remark that if we calculate the Laplacian $dd^\star + d^\star d$ of a one-form $\gamma_1 dx^1 + \gamma_2 dx^2$ we get with $\gamma = (\gamma_1, \gamma_2)$ that

$$\Delta\gamma = \nabla \operatorname{div}\gamma + \nabla^\perp\nabla^\perp\gamma. \tag{3.35}$$

Since this is a strongly elliptic operator, we can find a unique solution $\gamma \in W^{2,p}(B, \mathbb{R}^2)$ (by standard L^p-theory) of

$$\begin{aligned} \Delta\gamma &= \omega \quad \text{in} \quad B, \\ \gamma &= 0 \quad \text{on} \quad \partial B, \end{aligned} \tag{3.36}$$

Moreover we have the estimate

$$||\gamma||_{W^{2,p}(B)} \leq c||\omega||_{L^p(B)}. \tag{3.37}$$

Now an easy calculation (using (3.35)) shows that for every $h \in C^\infty(B, \mathbb{R}^2)$ we have

$$\begin{aligned} -\int_B \omega h &= \int_B \operatorname{div}\gamma \operatorname{div} h + \int_B \nabla^\perp\gamma\nabla^\perp h \\ &= -\int_B \Delta\gamma h + \int_{\partial B} \operatorname{div}\gamma h \cdot \nu + \int_{\partial B} \nabla^\perp\gamma \cdot \nu h. \end{aligned} \tag{3.38}$$

Since $\gamma = 0$ on ∂B we easily see that $\nabla^\perp\gamma \cdot \nu = 0$ on ∂B. Because we also know that $\Delta\gamma = \omega$ in B we get from (3.38) that

$$\operatorname{div}\gamma = 0 \tag{3.39}$$

on ∂B. Now we define $\alpha = \operatorname{div}\gamma$ and $\beta = \nabla^\perp\gamma$. This shows the first statement of the Lemma. For the second statement we note that (using (3.35) and (3.36))

$$0 = \sum_{i=1}^2 \nabla_i\omega_i = \Delta\alpha. \tag{3.40}$$

Therefore α is a harmonic function with zero boundary values (because of (3.39)) and hence vanishes identically. This shows that

$$\nabla^\perp\beta = \Delta\gamma = \omega$$

and therefore finishes the proof of the Lemma. □

Now we come to the main Theorem of this chapter. This theorem has been proved first by F. Hélein [17].

Theorem 3.3.2. *Let $u \in W^{1,2}(B, S^n)$ be a weakly harmonic map, then $u \in C^0(B, S^n)$.*

Proof. Let $u \in W^{1,2}(B, S^n)$ be a weakly harmonic map. Then we know from Lemma 3.1.2 that u is a weak solution of

$$\begin{aligned} -\Delta u^i &= \sum_j u^i \nabla u^j \nabla u^j \\ &= \sum_j \nabla u^j (u^i \nabla u^j - u^j \nabla u^i). \end{aligned} \tag{3.41}$$

By Theorem 3.1.3 we know that

$$\operatorname{div}(u^i \nabla u^j - u^j \nabla u^i) = 0 \tag{3.42}$$

weakly, for every $i, j \in \{1, \dots, n+1\}$. From Lemma 3.3.1 we get the existence of $B^{ij} \in W^{1,2}(B, \mathbb{R}^{n+1})$ such that

$$u^i \nabla u^j - u^j \nabla u^i = \nabla^\perp B^{ij}. \tag{3.43}$$

We insert this into (3.41) and get

$$\begin{aligned} -\Delta u^i &= \sum_j \nabla u^j \nabla^\perp B^{ij} \\ &= \sum_j \{u^j, B^{ij}\}. \end{aligned} \tag{3.44}$$

Now we are almost in a position to apply Wente's inequality but up to now we don't have that $u = 0$ on ∂B. To overcome this difficulty we decompose $u = v + h$, where $v \in W^{1,2}(B, \mathbb{R}^{n+1})$ is a solution of

$$\begin{aligned} -\Delta v^i &= \sum_j \{u^j, B^{ij}\} \quad \text{in} \quad B, \\ v &= 0 \quad \text{on} \quad \partial B, \end{aligned} \tag{3.45}$$

and h is the harmonic extension of u. From standard estimates for harmonic functions (see [14]) we get that h is continuous. The continuity of u now follows from the continuity of h and from an application of Wente's inequality, Theorem 3.2.1, to v. □

3.4 Higher regularity

In this section we follow the work of Chang, Wang & Yang [5] and prove that continuous solutions of two-dimensional geometric variational problems are as smooth as the data allows them to be. We consider weak solutions $u \in C^0 \cap W^{1,2}(B, \mathbb{R}^{n+1})$ of

$$\Delta u^i = f^i(x, \nabla u), \tag{3.46}$$

for every $i \in \{1, \dots, n+1\}$ and where $|f^i(x,p)| \le c(1 + |p|^2)$. We have the following Theorem.

Theorem 3.4.1. *Let $u \in C^0 \cap W^{1,2}(B, \mathbb{R}^{n+1})$ be a weak solution of* (3.46)*, then u is as smooth as the data permits.*

Proof. First of all we note that we can without loss of generality assume that for for every $i \in \{1, \dots, n+1\}$ and sufficiently small constants A and a we have

$$|f^i(x,p)| \leq A(1 + a|p|^2). \tag{3.47}$$

To see this we define for $r < 1$

$$u_1(x) = \frac{u(rx) - u(0)}{c(r)}, \tag{3.48}$$

where $c(r) = r + \sup_B |u(rx) - u(0)|$. Then it is easy to calculate that u_1 satisfies an equation of the form (3.46) with right hand side

$$f_1^i(x,p) = \frac{r^2}{c(r)} f^i(rx, \frac{c(r)}{r} p). \tag{3.49}$$

Thus

$$\begin{aligned}
|f_1(x,p)| =& \frac{r^2}{c(r)} |f(rx, \frac{c(r)}{r} p)| \\
\leq& \frac{cr^2}{c(r)} (1 + \frac{c(r)^2}{r^2} |p|^2) \\
\leq& c\sqrt{c(r)}(1 + \sqrt{c(r)}|p|^2)
\end{aligned}$$

and since u is continuous we get that $c(r) \to 0$ as $r \to 0$. This shows that we can assume (3.47) for u_1 and since smoothness results for u_1 imply directly the corresponding results for u this proves the claim.
Next we claim that if u is a solution of (3.46) in B, which satisfies (3.47) with $a|u| \leq c_1$ and $Ac_1 < \frac{1}{2}$, then we have

$$\int_{B_{\frac{1}{2}}} |\nabla u|^2 \leq c \int_B (1 + |u|^2), \tag{3.50}$$

where $c = c(c_1, A)$.
To prove this we consider a smooth cut-off function $\eta \in C_c^\infty(B, \mathbb{R})$ with $0 \leq \eta \leq 1$, $|\nabla \eta| \leq c$ and $\eta = 1$ in $B_{\frac{1}{2}}$. Then we choose $\eta^2 u$ as a test function in the weak formulation of (3.46) and get

$$\begin{aligned}
\int_B \eta^2 |\nabla u|^2 &= \int_B \nabla u \nabla(\eta^2 u) - 2\int_B u\eta \nabla u \nabla \eta \\
&= \int_B \eta^2 u f - 2\int_B u\eta \nabla u \nabla \eta \\
&\leq A\int_B \eta^2 |u| + Aa\int_B \eta^2 |u||\nabla u|^2 + \varepsilon \int_B \eta^2 |\nabla u|^2 + \frac{1}{\varepsilon}\int_B |u|^2 |\nabla \eta|^2 \\
&\leq (\frac{1}{2} + \varepsilon)\int_B \eta^2 |\nabla u|^2 + c(1 + \frac{1}{\varepsilon})\int_B (1 + |u|^2),
\end{aligned} \tag{3.51}$$

which proves the claim.
Next we need the following Lemma.

Lemma 3.4.2. *For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $A \leq \delta$, then for any solution u of* (3.46)*, satisfying* (3.47)*, $a|u| \leq c_1$ and $\frac{1}{|B|}\int_B |u|^2 \leq 1$, there exists a harmonic function $h : B_{\frac{1}{2}} \to \mathbb{R}^{n+1}$ such that*

$$\int_{B_{\frac{1}{2}}} |u-h|^2 \leq \varepsilon^2. \tag{3.52}$$

Proof. Assume that there exists some $\varepsilon > 0$ and sequences u_n and f_n such that $\frac{1}{|B|}\int_B |u_n|^2 \leq 1$, $f_n \leq \frac{1}{n}(1 + a|\nabla u_n|^2)$ and $\Delta u_n = f_n(x, \nabla u_n)$ but with

$$\int_{B_{\frac{1}{2}}} |u_n - v|^2 \geq \varepsilon^2 \tag{3.53}$$

for every harmonic function $v : B_{\frac{1}{2}} \to \mathbb{R}^{n+1}$. From (3.50) we get

$$\int_{B_{\frac{1}{2}}} |\nabla u_n|^2 \leq c,$$

which implies that, after choosing a subsequence, we have that there exists $u \in W^{1,2}(B_{\frac{1}{2}}, \mathbb{R}^{n+1})$ such that $\nabla u_n \rightharpoonup \nabla u$ weakly in L^2 and $u_n \to u$ strongly in L^2. Now we show that u is harmonic, which clearly contradicts (3.53). To see this we choose $\varphi \in C^\infty(B_{\frac{1}{2}}, \mathbb{R}^{n+1})$ and note that

$$\int \nabla\varphi \nabla u_n = \int \varphi f_n.$$

Letting $n \to \infty$ and using the estimate forf_n and u_n we get

$$\int \nabla\varphi\nabla u = 0,$$

which proves the Lemma. □

Next we have the following Corollary.

Corollary 3.4.3. *For any $0 < \gamma < 1$ and c_1, there exists $\varepsilon > 0$ and $0 < \lambda < \frac{1}{2}$, such that if $A \leq \varepsilon$ and u is a solution of* (3.46)*, satisfying* (3.47)*, $a|u| \leq 1$ and $\frac{1}{|B|}\int_B |u|^2 \leq 1$, then there exists a linear fucntion $l(x) = a_1 x + a_2$, with $|a_1| + |a_2| \leq c$, such that*

$$\frac{1}{|B_\lambda|}\int_{B_\lambda} |u-l|^2 \leq \lambda^{2(1+\gamma)}. \tag{3.54}$$

Proof. Let h be the harmonic function of Lemma 3.4.2 such that

$$\int_{B_{\frac{1}{2}}} |u-h|^2 \leq \varepsilon^2. \tag{3.55}$$

Using the assumptions we get that

$$\int_{B_{\frac{1}{2}}} |h|^2 \leq c.$$

From the energy estimates for harmonic functions we get

$$|h|(x) + |\nabla h|(x) + |\nabla^2 h|(x) \le c \int_{B_{\frac{1}{2}}} |h|^2 \le c, \tag{3.56}$$

for every $x \in B_{\frac{1}{4}}$. Now we let l be the first order Taylor polynomial of h in 0. Then we estimate for $\lambda \le \frac{1}{4}$

$$\begin{aligned} \frac{1}{|B_\lambda|} \int_{B_\lambda} |u - l|^2 &\le \frac{1}{|B_\lambda|} \int_{B_\lambda} |u - h|^2 + \frac{1}{|B_\lambda|} \int_{B_\lambda} |h - l|^2 \\ &\le c \frac{\varepsilon^2}{\lambda^2} + c\lambda^4, \end{aligned}$$

where we used (3.55) and (3.56) in the last line. Now we choose λ so small that $c\lambda^4 \le \frac{1}{2}\lambda^{2(1+\gamma)}$ and then we choose ε so that $c\frac{\varepsilon^2}{\lambda^2} \le \frac{1}{2}\lambda^{2(1+\gamma)}$. The bound on the coefficients of l follows from (3.56). □

Next we iterate this corollary to get.

Lemma 3.4.4. *For any $0 < \gamma < 1$ there exists $\varepsilon > 0$, $c > 0$ and $0 < \lambda < \frac{1}{2}$ such that if u is a solution of (3.46) satisfying (3.47) with $A \le \varepsilon$ and $|u| \le 1$, then there exist linear functions $l_k(x) = B_k x + C_k$, $k \in \mathbb{N}_0$, satisfying*

$$\lambda^k |B_k - B_{k+1}| + |C_k - C_{k+1}| \le c\lambda^{(\gamma+1)k}, \tag{3.57}$$

such that

$$\frac{1}{|B_{\lambda^k}|} \int_{B_{\lambda^k}} |u - l_k|^2 \le 2\lambda^{2(\gamma+1)k}. \tag{3.58}$$

Proof. We prove this statement by induction. The case $k = 0$ follows from the assumptions on u by choosing $B_0 = C_0 = 0$ and the case $k = 1$ follows from Corollary 3.4.3 by choosing $B_1 = a_1$ and $C_1 = a_2$.
Now we assume that the statement of the Lemma has been verified up to order k. First we observe from (3.57) and $B_0 = 0$ that

$$\begin{aligned} |\nabla l_k| = |B_k| &\le \sum_{i=0}^{k-1} |B_{i+1} - B_i| \\ &\le c \sum_{i=0}^{\infty} \lambda^{\gamma k} \\ &\le \frac{c}{1 - 2^{-\gamma}}. \end{aligned}$$

The same estimate holds true for $|C_k|$. Now we define $w(x) = \frac{(u - l_k)(\lambda^k x)}{\lambda^{(1+\gamma)k}}$ and an easy calculation shows that

$$\Delta w^i = f_1^i(x, \nabla w), \tag{3.59}$$

where

$$f_1(x,p) = \lambda^{(1-\gamma)k} f(\lambda^k x, \lambda^{\gamma k} p + \nabla l_k).$$

By the assumption (3.47) we see that

$$|f_1(x,p)| \leq A\lambda^{(1-\gamma)k}(1 + 2a|\nabla l_k|^2) + Aa\lambda^{(1+\gamma)k}|p|^2. \tag{3.60}$$

Now we claim that we can apply Corollary 3.4.3 to w. To see this we estimate

$$2|w|(x)\lambda^{(1+\gamma)k} = 2|(u - l_k)|(\lambda^k x) \leq 2(1 + \lambda^k|\nabla l_k| + |C_k|) \leq \frac{c}{1 - 2^{-\gamma}}$$

and

$$\frac{1}{|B|}\int_B |w|^2 = \lambda^{-2(1+\gamma)k}\frac{1}{|B_{\lambda^k}|}\int_{B_{\lambda^k}} |u - l_k|^2 \leq 1.$$

This shows that we can apply Corollary 3.4.3 to w and we get the existence of a linear function $l(x) = Bx + C$, with $|B| + |C| \leq c$, such that

$$\frac{1}{|B_\lambda|}\int_{B_\lambda} |w - l|^2 \leq \lambda^{2(1+\gamma)}. \tag{3.61}$$

Now we define $l_{k+1}(x) = l_k(x) + \lambda^{(1+\gamma)k} l(\frac{x}{\lambda^k})$ and we check that $B_{k+1} = B_k + B\lambda^{\gamma k}$, $C_{k+1} = C_k + C\lambda^{(1+\gamma)k}$ and

$$\begin{aligned}\frac{1}{|B_{\lambda^{k+1}}|}\int_{B_{\lambda^{k+1}}} |u - l_{k+1}|^2 =& \frac{1}{|B_{\lambda^{k+1}}|}\int_{B_{\lambda^{k+1}}} |\lambda^{(1+\gamma)k}(w(\frac{x}{\lambda^k}) - l(\frac{x}{\lambda^k})|^2 \\ =& \lambda^{2(1+\gamma)k}\frac{1}{|B_\lambda|}\int_{B_\lambda} |w - l|^2 \\ \leq& \lambda^{2(1+\gamma)k + 2(1+\gamma)} \\ =& \lambda^{2(1+\gamma)(k+1)}.\end{aligned}$$

This finishes the proof of the induction step and therefore the proof of the Lemma. □

To finish the proof of Theorem 3.4.1 we note that because of (3.57) B_k and C_k converge exponentially to some B respectively C and if we then denote $l(x) = Bx + C$ we have

$$\frac{1}{|B_{\lambda^k}|}\int_{B_{\lambda^k}} |u - l|^2 \leq 2\lambda^{2(1+\gamma)k} \tag{3.62}$$

for every $k \in \mathbb{N}$. Therefore $u \in C^{1,\gamma}$ by classical estimates in Campanato-, resp. Morrey-spaces (see e.g. [13]) and higher reguarity then follows from Schauder theory. □

Chapter 4

Hardy space

In this chapter we study the Hardy space $\mathcal{H}^1$. In particular we prove the atomic decomposition of $\mathcal{H}^1$. An easy consequence of this will be that the L^p-theory (see Theorem 2.3.3) remains true if the right hand side $f \in \mathcal{H}^1$.

4.1 Maximal function and a Theorem of Müller

Before coming to Hardy spaces we will prove some more or less standard results for the maximal function. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ we let for $\alpha \geq 0$

$$\lambda_f(\alpha) = |\{x \in \mathbb{R}^n | |f(x)| > \alpha\}| \tag{4.1}$$

be the distribution function of f.

Lemma 4.1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable and $0 < p < \infty$, then we have*

$$||f||_{L^p}^p = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha. \tag{4.2}$$

Proof. We have

$$|f(x)|^p = p \int_0^{|f(x)|} \alpha^{p-1} d\alpha = p \int_0^\infty \alpha^{p-1} \chi_{\{\alpha | \alpha < |f(x)|\}} d\alpha.$$

Using Fubini‘s theorem this gives

$$||f||_{L^p}^p = p \int_0^\infty \alpha^{p-1} \Big(\int_{\mathbb{R}^n} \chi_{\{x | |f(x)| > \alpha\}} dx \Big) d\alpha = p \int_0^\infty \alpha^{p-1} \lambda_f(\alpha) d\alpha.$$

□

Next we define the maximal function.

Definition 4.1.2. *For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define its maximal function $Mf : \mathbb{R}^n \to \mathbb{R}$ by*

$$Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x)| dx. \tag{4.3}$$

Before coming to the Hardy-Littlewood Theorem for the maximal function we note without proof the covering Lemma of Besicovitch.

Lemma 4.1.3. *Let $E \subset \mathbb{R}^n$ be measurable and suppose that $E \subset \cup_j B_j$, where the family $\{B_j\}_{j \in J}$ consists of balls with bounded diameter (i.e. $\sup_j \operatorname{diam}(B_j) < \infty$). Then there exists a countable disjoint subfamily $\{B_{j_k}\}_{k \in \mathbb{N}}$ such that $E \subset \cup_k 5B_{j_k}$ and*

$$|E| \leq 5^n \sum_{k=1}^{\infty} |B_{j_k}|. \tag{4.4}$$

Now we can prove the so called Hardy-Littlewood theorem for the maximal function.

Theorem 4.1.4. *Let $1 < p \leq \infty$ and $f \in L^p(\mathbb{R}^n)$. Then we have*

$$||M(f)||_{L^p} \leq c||f||_{L^p}, \tag{4.5}$$

where $c = c(p, n)$. Moreover for $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$ we have

$$|\{x \in \mathbb{R}^n | Mf(x) > \alpha\}| \leq \frac{c}{\alpha}||f||_{L^1}, \tag{4.6}$$

where $c = c(n)$.

Proof. First we prove (4.6). We let $E^\alpha_{Mf} = \{x \in \mathbb{R}^n | Mf(x) > \alpha\}$. By definition, for any $x \in E^\alpha_{Mf}$ there exists a ball B^x of radius $r_x > 0$ such that

$$\int_{B^x} |f(y)| dy > \alpha |B^x|. \tag{4.7}$$

Now we fix $R > 0$ and let $A_R = \cup_{\{x \in E^\alpha_{Mf} | 0 < r_x < R\}} B^x$. We clearly have $A_R \subset A_S$ for $R < S$, $E^\alpha_{Mf} \subset \cup_{R=1}^\infty A_R$ and the balls in A_R have bounded diameter. Therefore we can apply Lemma 4.1.3 to get the existence of a pairwise disjoint subfamily $B_R = \{B^{x_k}\}_{k \in \mathbb{N}}$ of $\cup_{\{x \in E^\alpha_{Mf} | 0 < r_x < R\}} B^x$ such that

$$\begin{aligned} A_R &\subset \cup_k 5B^{x_k} \\ |A_R| &\leq 5^n \sum_k |B^{x_k}|. \end{aligned} \tag{4.8}$$

For $R \to \infty$ we conclude

$$\begin{aligned} \alpha |E^\alpha_{Mf}| &= \lim_{R \to \infty} \alpha |A_R| \\ &\leq \lim_{R \to \infty} 5^n \sum_k \alpha |B^{x_k}| \\ &\leq 5^n \lim_{R \to \infty} \sum_k \int_{B^{x_k}} |f(y)| dy \\ &\leq 5^n ||f||_{L^1}, \end{aligned} \tag{4.9}$$

where we used (4.7) and the properties of the subfamily B_R. This shows (4.6). Now we turn to the proof of (4.5). First we note that the case $p = \infty$ is trivial with $c(n, \infty) = 1$ and hence

we assume from now on $1 < p < \infty$.
For $\alpha > 0$ we let

$$f_1(x) = \begin{cases} f(x) & \text{if} |f(x)| \geq \frac{\alpha}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have for all $x \in \mathbb{R}^n$

$$\begin{aligned} |f(x)| &\leq |f_1(x)| + \frac{\alpha}{2} \quad \text{and} \\ |Mf(x)| &\leq |Mf_1(x)| + \frac{\alpha}{2}. \end{aligned}$$

Therefore we get

$$E^{\alpha}_{Mf} \subset E^{\frac{\alpha}{2}}_{Mf_1}. \tag{4.10}$$

Since $f_1 \in L^1(\mathbb{R}^n)$ we can apply (4.6) and get

$$|E^{\frac{\alpha}{2}}_{Mf_1}| \leq \frac{2c}{\alpha}||f_1||_{L^1}. \tag{4.11}$$

Going back to f (by using (4.10) and (4.11)) we get

$$\begin{aligned} |E^{\alpha}_{Mf}| \leq& |E^{\frac{\alpha}{2}}_{Mf_1}| \\ \leq& \frac{2c}{\alpha}||f_1||_{L^1} \\ =& \frac{2c}{\alpha}\int_{\{x||f(x)|\geq\frac{\alpha}{2}\}} |f(x)|dx. \end{aligned} \tag{4.12}$$

Using Lemma 4.1.1 and (4.12) we calculate

$$\begin{aligned} ||Mf||^p_{L^p} &= p\int_0^\infty \alpha^{p-1}|E^{\alpha}_{Mf}|d\alpha \\ &\leq 2cp\int_0^\infty \alpha^{p-2}(\int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n||f(x)|\geq\frac{\alpha}{2}\}}|f(x)|dx)d\alpha. \end{aligned} \tag{4.13}$$

By Fubini‘s theorem we get

$$\begin{aligned} ||Mf||^p_{L^p} \leq& 2cp\int_{\mathbb{R}^n} |f(x)|(\int_0^{2|f(x)|} \alpha^{p-2}d\alpha)dx \\ =& \frac{2^p cp}{p-1}\int_{\mathbb{R}^n} |f(x)|^p. \end{aligned} \tag{4.14}$$

This finishes the proof of the Theorem. □

Remark 4.1.5. *It is easy to see that* (4.5) *can not be extended to* $p = 1$*. Namely if* $f \in L^1(\mathbb{R}^n)$ *and* $|f|(x) \neq 0$ *for all* $x \in \mathbb{R}^n$ *then* Mf *is not in* L^1*. To see this we let* $\varepsilon > 0$ *be small and choose* $r_0 > 0$ *such that*

$$\int_{B_{r_0}} |f(x)|dx \geq \varepsilon.$$

Now for $|x| > r_0$ *we have* $B_{r_0} \subset B_{2|x|}(x)$ *and therefore*

$$\begin{aligned} Mf(x) =& \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(x)|dx \\ \geq& \frac{1}{|B_{2|x|}(x)|} \int_{B_{2|x|}(x)} |f(x)|dx \\ =& \frac{c\varepsilon}{|x|^n} \\ \notin& L^1(\mathbb{R}^n). \end{aligned}$$

This shows that we can not expect to have global integrability of Mf *if* $f \in L^1(\mathbb{R}^n)$. *But there are also counterexamples to local integrabilty. Namely let* $f : \mathbb{R}^n \to \mathbb{R}$ *be given by*

$$f(x) = \frac{1}{|x|^n |\log(|x|)|^2} \chi_{B_1}.$$

An easy calculation, using polar coordinates, shows that for every $t < 1$ *we have*

$$\begin{aligned} \int_{B_t} f(x)dx =& \int_0^t \frac{r^{n-1}}{r^n |\log r|^2} dr \\ =& \int_{|\log t|}^{\infty} \frac{1}{s^2} ds \\ =& \frac{1}{|\log t|}, \end{aligned}$$

where we used the substitution $s = |\log r|$ *in the second line. This shows that* $f \in L^1(B_{\frac{1}{2}})$ *and moreover for every* $x \in B_{\frac{1}{2}}$ *we have*

$$\begin{aligned} Mf(x) \geq& \frac{1}{2^n |x|^n} \int_{B_{2|x|}(0)} f(x)dx \\ \geq& \frac{c}{|x|^n \log(2|x|)} \\ \notin& L^1(B_{\frac{1}{2}}). \end{aligned}$$

In the next lemma we see that if we assume slighly more than being in L^1 the Hardy-Littlewood theorem remains true.

Lemma 4.1.6. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded and let* $f \in L^1 \log L^1(\Omega)$, *i.e.*

$$\int_{\Omega} |f(x)| \log^+ |f(x)|dx < \infty, \tag{4.15}$$

where $\log^+ |f(x)| = \max\{0, \log |f(x)|\}$, *then we have* $Mf \in L^1(\Omega)$.

Proof. From Lemma 4.1.1 we know that

$$||Mf||_{L^1(\Omega)} \leq 2 \int_0^{\infty} |E^{2\alpha}_{\chi_\Omega Mf}| d\alpha,$$

and hence

$$||Mf||_{L^1(\Omega)} \leq 2|\Omega| + 2\int_1^\infty |E^{2\alpha}_{\chi_\Omega Mf}|d\alpha. \tag{4.16}$$

Arguing as in the proof of (4.12) in the previous theorem, we have

$$\begin{aligned}\int_1^\infty |E^{2\alpha}_{\chi_\Omega Mf}|d\alpha \leq & \int_1^\infty (\frac{c}{\alpha}\int_{\mathbb{R}^n} \chi_{\{x\in\Omega||f(x)|\geq\alpha\}}|f(x)|dx)d\alpha \\ =& c\int_\Omega |f(x)|(\int_1^{\max(1,|f(x)|)} \frac{d\alpha}{\alpha})dx \\ \leq & c\int_\Omega |f(x)|\log^+|f(x)|dx. \end{aligned} \tag{4.17}$$

Combining this with (4.16) finishes the proof. □

Next we prove the Calderon-Zygmund decomposition result.

Theorem 4.1.7. *Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and let $\alpha > 0$. Then there exists a sequence of disjoint cubes $(C_k)_{k\in\mathbb{N}}$ such that*

(i) The average of f on all cubes is bounded from below and above by

$$\alpha < \frac{1}{|C_k|}\int_{C_k} f(x)dx \leq 2^n\alpha. \tag{4.18}$$

(ii) On $\Omega^c = (\cup_{k=1}^\infty C_k)^c$ we have a.e.

$$f(x) \leq \alpha. \tag{4.19}$$

(iii) There exists $c = c(n)$ such that

$$|\Omega| \leq \frac{c}{\alpha}||f||_{L^1(\mathbb{R}^n)}. \tag{4.20}$$

Proof. We divide $\mathbb{R}^n$ into equal cubes such that the volume of them is larger than or equal to $\frac{||f||_{L^1}}{\alpha}$. This means that for every cube C_0 in this decomposition we have

$$\frac{1}{|C_0|}\int_{C_0} f(x)dx \leq \alpha. \tag{4.21}$$

Now we fix one cube C_0 in this decomposition and decompose it into 2^n equal disjoint cubes with half of the side length. For these new cubes there are now two possibilities: either (4.21) still holds, these cubes are called good cubes and are denoted by C_1^g, or (4.21) does not hold anymore. These cubes are called bad cubes and are denoted by C_1^b. In the next step we decompose each cube in C_1^g into equal disjoint cubes with half side-length and we leave C_1^b unchanged. The good cubes of this decomposition we be called C_2^g and the bad ones C_2^b. If we continue to repeat this process we finally can define $\Omega = \cup_{k=1}^\infty C_k^b$ as the union of all cubes which violate the estimate (4.21) in some step of the decomposition process.
Next we note that if $C_i^b \in \Omega$ then

$$\frac{1}{|C_i^b|}\int_{C_i^b} f(x)dx > \alpha. \tag{4.22}$$

But we also have $C_i^b \subset C_{i-1}^g$ and therefore $2^n|C_i^b| = |C_{i-1}^g|$. Combining this with $f \geq 0$ and (4.22) gives

$$\alpha < \frac{1}{|C_i^b|}\int_{C_i^b} f(x)dx \leq \frac{2^n}{|C_{i-1}^g|}\int_{C_{i-1}^g} f(x)dx \leq 2^n\alpha, \tag{4.23}$$

which proves (i).
On Ω^c we can apply Lebesgue's differentiation theorem to get for almost every $x \in \Omega^c$

$$f(x) = \lim_{i\to\infty} \frac{1}{|C_{i,x}^g|}\int_{C_{i,x}^g} f(x)dx \leq \alpha, \tag{4.24}$$

where $C_{i,x}^g$ denotes the cube in C_i^g which contains x. This shows (ii).
To see (iii) we use (4.22) to get

$$|\Omega| = \sum_{k=1}^{\infty} |C_k^b| < \frac{1}{\alpha}\int_{\Omega} f(x)dx \leq \frac{1}{\alpha}||f||_{L^1}. \tag{4.25}$$

□

With the help of this Calderon-Zygmund decomposition we can now show the opposite estimate to Lemma 4.1.6.

Lemma 4.1.8. *Let $f \in L^1(\mathbb{R}^n)$ be supported in a compact set B. Then we have that $Mf \in L^1(B)$ if and only if $f \in L^1 \log L^1(B)$.*

Proof. Because of Lemma 4.1.6 it remains to show that $Mf \in L^1(B)$ implies $f \in L^1 \log L^1(B)$. To show this we first prove that

$$|\{x|Mf(x) > c\alpha\}| \geq \frac{1}{2^n\alpha}\int_{\{x||f(x)|>\alpha\}} |f(x)|dx, \tag{4.26}$$

for some appropriate constant $c > 0$. To see this we apply Theorem 4.1.7 to $|f| \geq 0$ and $\alpha > 0$. From this we get cubes C_k such that

$$2^n\alpha \geq \frac{1}{|C_k|}\int_{C_k} |f(x)|dx \geq \alpha. \tag{4.27}$$

Thus if $x \in C_k$ we have that $Mf(x) > c\alpha$, where the constant c is needed to pass from balls to cubes in the definition of Mf. As a direct consequence of this and (4.27) we have

$$\begin{aligned} |\{x|Mf(x) > c\alpha\}| \geq \sum_{k=1}^{\infty} |C_k| \geq & \frac{1}{2^n\alpha}\int_{\Omega} |f(x)|dx \\ \geq & \frac{1}{2^n\alpha}\int_{\{x||f(x)|>\alpha\}} |f(x)|dx, \end{aligned} \tag{4.28}$$

since $\{x||f(x)| > \alpha\} \subset \Omega$, which follows from (ii) in Theorem 4.1.7.
Now we estimate with the help of Fubini‘s Theorem, Lemma 4.1.1 and (4.26)

$$\begin{aligned}||Mf||_{L^1} &= \int_0^\infty \lambda_{Mf}(\alpha)d\alpha \\ &\geq \int_1^\infty |\{x|Mf(x) > \alpha\}|d\alpha \\ &\geq c\int_1^\infty \frac{1}{\alpha}(\int_{\{x||f(x)|>\alpha\}} |f(x)|dx)d\alpha \\ &= c\int_B |f(x)|(\int_1^{\max(1,|f(x)|)} \alpha^{-1}d\alpha)dx \\ &= c\int_B |f(x)|\log^+|f(x)|dx,\end{aligned}$$

which proves the Lemma. □

In the next Lemma we prove that the divergence of the cofactor matrix of ∇u is equal to zero. This was first observed by Morrey [23] (Lemma 4.6.4). Our proof is taken from [30].

Lemma 4.1.9. *Let $u \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then we have*

$$\sum_{i=1}^n \partial_i(\operatorname{cof}\nabla u)_i^k = 0, \tag{4.29}$$

for every $1 \leq k \leq n$.

Proof. For every matrix P we have the formula $(\det P)id = P^T \operatorname{cof} P$ and therefore

$$\det P\delta_{ij} = \sum_{k=1}^n P_i^k(\operatorname{cof} P)_j^k. \tag{4.30}$$

This implies for $r, s \in \{1, \ldots, n\}$ (choose $i = j = s$)

$$\begin{aligned}\frac{\partial \det P}{\partial P_s^r} =& \sum_{k=1}^n (\frac{\partial P_s^k}{\partial P_s^r}(\operatorname{cof} P)_s^k + P_s^k \frac{\partial(\operatorname{cof} P)_s^k}{\partial P_s^r}) \\ =& (\operatorname{cof} P)_s^r.\end{aligned} \tag{4.31}$$

Inserting $P = \nabla u$ into (4.30), differentiating this with respect to x_j and adding we obtain

$$\sum_{j,r,s=1}^n \delta_{ij}(\operatorname{cof}\nabla u)_s^r \partial_j\partial_s u^r = \sum_{r,j=1}^n (\partial_j\partial_i u^r(\operatorname{cof}\nabla u)_j^r + \partial_i u^r \partial_j(\operatorname{cof}\nabla u)_j^r), \tag{4.32}$$

where we also used (4.31). Since the term on the left and the first term on the right cancel each other, we end up with

$$\sum_{r=1}^n \partial_i u^r (\sum_{j=1}^n \partial_j(\operatorname{cof}\nabla u)_j^r) = 0, \tag{4.33}$$

for every $1 \le i \le n$. This means that the vector $(\sum_{j=1}^n \partial_j (\operatorname{cof} \nabla u)_j^r)$ is a solution of the linear system $A^T y = 0$, where $A = \nabla u$. If we now have a point $x_0 \in \mathbb{R}^n$ for which $\det \nabla u(x_0) \neq 0$, then we can conclude that

$$(\sum_{j=1}^n \partial_j (\operatorname{cof} \nabla u)_j^r)(x_0) = 0. \tag{4.34}$$

If on the other hand $\det \nabla u(x_0) = 0$ then we choose $\varepsilon > 0$ such that $\det(\nabla u + \varepsilon id) \neq 0$ and we do the same calculation for $\tilde{u} = u + \varepsilon x$. Finally we let $\varepsilon \to 0$ and finish the proof of the Lemma. □

Next we use degree theory to prove an isoperimetric inequality, which is taken from [24].

Proposition 4.1.10. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u \in W^{1,n}(\Omega, \mathbb{R}^n)$, $x \in \Omega$ and $R < \operatorname{dist}(x, \partial\Omega)$. Then we have for a.e. $r \in (0, R)$*

$$|\int_{B_r(x)} \det \nabla u(y) dy|^{\frac{n-1}{n}} \le c \int_{\partial B_r(x)} |\operatorname{adj} \nabla u| dS, \tag{4.35}$$

where $\operatorname{adj} \nabla u = (\operatorname{cof} \nabla u)^T$ is the adjungate of ∇u.

Proof. As already mentioned we use degree theory for the proof of this result. Good references for the degree theory are for example [25], [10], [8], [30], etc.
First of all we would like to mention that it suffices to prove the Proposition for $u \in C^\infty(\Omega, \mathbb{R}^n)$. Indeed, for any $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ there exists a sequence $u_j \in C^\infty(\Omega, \mathbb{R}^n)$ such that $u_j \to u$ in $W^{1,n}(B_r(x), \mathbb{R}^n)$ and such that for a.e. $r \in (0, R)$, $u_j \to u$ in $W^{1,n}(\partial B_r(x), \mathbb{R}^n)$. To see the last statement consider the function $h_j(r) = \int_{\partial B_r(x)} (|\nabla u_j - \nabla u|^n + |u_j - u|^n) dS$. We have that $\lim_{j\to\infty} \int_0^R h_j(r) dr = 0$.
By this argument we can now assume that $u \in C^\infty(\Omega, \mathbb{R}^n)$ and we fix some $0 < r < R$. We let $y_0 \in \mathbb{R}^n \backslash u(\partial B_r(x))$ and denote by Γ the connected component of $\mathbb{R}^n \backslash u(\partial B_r(x))$ which contains y_0. Moreover we let $f \in C_c^\infty(\Gamma)$ such that $\int_{\mathbb{R}^n} f(y) dy = 1$. We can then define the degree of $u|_{B_r(x)}$ at y_0 by

$$\deg(u, B_r(x), y_0) = \int_{B_r(x)} f(u(y)) \det \nabla u(y) dy. \tag{4.36}$$

The degree does not depend on the choice of f, is integer-valued and the function $y \to \deg(u, B_r(x), y)$ is constant on each connected component of $\mathbb{R}^n \backslash u(\partial B_r(x))$. From this it follows (by decomposing $\mathbb{R}^n$ into connected components of $\mathbb{R}^n \backslash u(\partial B_r(x))$) that for any $g \in C^0(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} g(y) \deg(u, B_r(x), y) dy = \int_{B_r(x)} g(u(y)) \det \nabla u(y) dy. \tag{4.37}$$

The integral on the left hand side is well-defined since the degree is defined up to a set of measure zero. Next we calculate for every $v \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$

$$\begin{aligned} \partial_j \big(v^i(u(x))(\operatorname{adj} \nabla u)_i^j\big) =& \sum_k \big((\partial_k v^i)(u(x)) \partial_j u^k (\operatorname{adj} \nabla u)_i^j\big) + v^i(u(x)) \partial_j (\operatorname{adj} \nabla u)_i^j\big) \\ =& (\operatorname{div} v)(u(x)) \det \nabla u, \end{aligned} \tag{4.38}$$

where we used the fact that $(\det \nabla u)id = \nabla u(\operatorname{adj} \nabla u) = \nabla u(\operatorname{cof} \nabla u)^T$ and Lemma 4.1.9. Integrating this over $B_r(x)$ yields for every $v \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$

$$\int_{B_r(x)} (\operatorname{div} v)(u(y)) \det \nabla u(y) dy = \int_{\partial B_r(x)} v^i(u(x))(\operatorname{adj} \nabla u)^j_i \nu_j dS, \tag{4.39}$$

where ν is the outer unit normal. Combining (4.37) and (4.39) we see that

$$|\int_{\mathbb{R}^n} (\operatorname{div} v)(y) \deg(u, B_r(x), y) dy| \le ||v||_{C^0(\mathbb{R}^n)} \int_{\partial B_r(x)} |\operatorname{adj} \nabla u| dS. \tag{4.40}$$

This shows that $\deg(u, B_r(x), \cdot) \in BV(\mathbb{R}^n)$ (for the definition of the BV-space see [15]) and that

$$\int_{\mathbb{R}^n} |D \deg(u, B_r(x), \cdot)| \le \int_{\partial B_r(x)} |\operatorname{adj} \nabla u| dS. \tag{4.41}$$

The Sobolev inequality for BV-functions (see [15], Theorem 1.28) yields

$$|| \deg(u, B_r(x), \cdot)||_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \le c \int_{\partial B_r(x)} |\operatorname{adj} \nabla u| dS. \tag{4.42}$$

Since $\deg(u, B_r, \cdot)$ is integer-valued we have that $|\deg(u, B_r(x), y)| \le |\deg(u, B_r(x), y)|^{\frac{n}{n-1}}$ and therefore we get from (4.37) (with $g \equiv 1$) and (4.42)

$$\begin{aligned} |\int_{B_r(x)} \det \nabla u(y) dy| \le& \int_{\mathbb{R}^n} |\deg(u, B_r(x), y)| dy \\ \le& \int_{\mathbb{R}^n} |\deg(u, B_r(x), y)|^{\frac{n}{n-1}} dy \\ \le& c(\int_{\partial B_r(x)} |\operatorname{adj} \nabla u| dS)^{\frac{n}{n-1}}, \end{aligned} \tag{4.43}$$

which proves the Proposition. □

Now we prove a Theorem of Müller [24] on the higher integrability of the determinant.

Theorem 4.1.11. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be bounded and open and let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be such that $\det(\nabla u) \ge 0$ almost everywhere. Then we have for every compact set $K \subset \Omega$ that $\det(\nabla u) \log(2 + \det(\nabla u)) \in L^1(K)$ and moreover*

$$|| \det(\nabla u) \log(2 + \det(\nabla u))||_{L^1(K)} \le c(K, ||u||_{W^{1,n}(\Omega)}). \tag{4.44}$$

Proof. We fix $K \subset \Omega$ and let $g = \chi_K \det(\nabla u)$. By Lemma 4.1.8 we only have to show that

$$||Mg||_{L^1(B)} \le c(K, ||u||_{W^{1,n}(\Omega)}). \tag{4.45}$$

where $\Omega \subset B$ is some ball. Now we let $d = \operatorname{dist}(K, \partial\Omega)$ and we note that

$$\begin{aligned} Mg(x) =& \sup_{t>0} \frac{1}{|B_t(x)|} \int_{B_t(x)} |g(y)| dy \\ \le& \sup_{0<t<\frac{d}{4}} \frac{1}{|B_t(x)|} \int_{B_t(x)} |g(y)| dy + \sup_{t \ge \frac{d}{4}} \frac{1}{|B_t(x)|} \int_{B_t(x)} |g(y)| dy \\ \le& \sup_{0<t<\frac{d}{4}} \frac{1}{|B_t(x)|} \int_{B_t(x)} |g(y)| dy + c(d)||u||^n_{W^{1,n}(\Omega)}. \end{aligned} \tag{4.46}$$

Now we denote $A := \{x \in \Omega |\operatorname{dist}(x, \partial\Omega) > \frac{d}{2}\}$ and estimate

$$||Mg||_{L^1(B)} = ||Mg||_{L^1(A)} + ||Mg||_{L^1(B\setminus A)}.$$

By (4.46) we get

$$||Mg||_{L^1(B\setminus A)} \leq c(d,B)||u||^n_{W^{1,n}(\Omega)}. \tag{4.47}$$

Therefore we see that it remains to estimate the L^1-norm of

$$\sup_{0<t<\frac{d}{4}} \frac{1}{|B_t(x)|}\int_{B_t(x)\cap A} |g(x)|dx. \tag{4.48}$$

To this end we use the fact that $\det(\nabla u) \geq 0$ and Proposition 4.1.10 to estimate for a.e. $r \in (t, 2t)$

$$(\int_{B_t(x)} |g(y)|dy)^{\frac{n-1}{n}} \leq (\int_{B_r(x)} \det(\nabla u)(y)dy)^{\frac{n-1}{n}} \leq c\int_{\partial B_r(x)} \operatorname{adj}(\nabla u)(y)dy. \tag{4.49}$$

Now we integrate this inequality over r from t to $2t$ and divide by $t^n|B_1|^{\frac{n-1}{n}}$ to get

$$(\frac{1}{|B_t(x)|}\int_{B_t(x)} |g(y)|dy)^{\frac{n-1}{n}} \leq c\frac{1}{|B_{2t}(x)|}\int_{B_{2t}(x)} |\operatorname{adj}(\nabla u)(y)dy| \leq M(\chi_\Omega \operatorname{adj}(\nabla u)). \tag{4.50}$$

Altogether this yields with the help of Theorem 4.1.4

$$\begin{aligned} ||Mg||_{L^1(B)} \leq & c(d,B,||u||^n_{W^{1,n}(\Omega)}) + c||M(\chi_\Omega \operatorname{adj}(\nabla u))||^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}} \\ \leq & c(d,B,||u||^n_{W^{1,n}(\Omega)}) + c||\operatorname{adj}(\nabla u)||^{\frac{n}{n-1}}_{L^{\frac{n}{n-1}}(\Omega)} \\ \leq & c(d,B,||u||^n_{W^{1,n}(\Omega)}), \end{aligned} \tag{4.51}$$

which proves the Theorem. □

4.2 Hardy space

In this section we introduce the Hardy space, derive some useful properties of it and give some examples of functions in the Hardy space.
We begin by defining a class of test functions T as follows

$$T = \{\phi \in C^\infty(\mathbb{R}^n)|\operatorname{spt}\phi \subset B \quad \text{and} \quad ||\nabla\phi||_{L^\infty} \leq 1\}. \tag{4.52}$$

For every $\phi \in T$ it is easy to see that $||\phi||_{L^\infty} \leq 1$. Next, for every $t > 0$, we define

$$\phi_t(x) = t^{-n}\phi(\frac{x}{t}). \tag{4.53}$$

We have $\operatorname{spt}\phi_t \subset B_t$ and $||\nabla\phi_t||_{L^\infty} \leq ct^{-(n+1)}$.

Definition 4.2.1. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We define the grand maximal function of f by*

$$f^\star(x) = \sup_{\phi\in T}\sup_{t>0} |\phi_t \star f(x)|. \tag{4.54}$$

Remark 4.2.2. *It is easy to see (Exercise!) that if we replace the condition $||\nabla\phi||_{L^\infty} \le 1$ in the definition of T by the requirement that $||\nabla\phi||_{L^\infty} \le c$ then the new grand maximal function is equivalent to the one defined above.*

With the help of the grand maximal function we can define the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ as follows.

Definition 4.2.3. *We say that $f \in L^1(\mathbb{R}^n)$ lies in the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ if $f^\star \in L^1(\mathbb{R}^n)$. Moreover the Hardy norm is defined by*

$$||f||_{\mathcal{H}^1} = ||f^\star||_{L^1}. \tag{4.55}$$

Remark 4.2.4. *See [9] and [33] for other possible definitions of $\mathcal{H}^1$ and a proof of the equivalence of all these definitions.*

In the next Lemma we derive an useful estimate for the grand maximal function.

Lemma 4.2.5. *For every $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we have that*

$$f^\star(x) \le cMf(x), \tag{4.56}$$

for every $x \in R^n$. If moreover $f \ge 0$ we have additionally that

$$Mf(x) \le cf^\star(x), \tag{4.57}$$

for every $x \in \mathbb{R}^n$.

Proof. We calculate

$$\begin{aligned} f^\star(x) &= \sup\nolimits_{\phi\in T}\sup\nolimits_{t>0} |\int_{\mathbb{R}^n} \phi_t(x-y)f(y)dy| \\ &= \sup\nolimits_{\phi\in T}\sup\nolimits_{t>0} |\int_{\mathbb{R}^n} t^{-n}\phi(\frac{x-y}{t})f(y)dy| \\ &\le \sup\nolimits_{\phi\in T}\sup\nolimits_{t>0} \frac{||\phi||_{L^\infty}}{t^n}\int_{B_t(x)} |f(y)|dy \\ &\le c\sup\nolimits_{t>0} t^{-n}\int_{B_t(x)} |f(y)|dy \\ &\le cMf(x). \end{aligned}$$

This proves (4.56). To show (4.57) we fix a positive function $\phi \in T$ such that $\phi(x) = \frac{1}{4}$ for every $x \in B_{\frac{1}{2}}$. Then it is easy to check that

$$\phi_{2t}(x) \ge 2^{-(n+2)}t^{-n}\chi_{B_t}(x).$$

Using this and the positivity of f we calculate

$$\begin{aligned} Mf(x) &= \omega_n \sup\nolimits_{t>0} t^{-n}\int_{B_t(x)} f(y)dy \\ &\le c\sup\nolimits_{t>0}\int_{B_t(x)} \phi_{2t}(x-y)f(y)dy \\ &\le c\sup\nolimits_{\phi\in T}\sup\nolimits_{t>0} |\phi_t \star f(x)| \\ &= cf^\star(x). \end{aligned}$$

This proves (4.57). □

The next Lemma is a technical result which we need later on.

Lemma 4.2.6. *Let $f \in L^1(\mathbb{R}^n)$ with $\operatorname{spt} f \subset B_R$. Then if $|x| \geq 2R$ we have that*

$$f^\star(x) = \sup_{\phi\in T} \sup_{t>0} |\phi_t \star f(x)| = \sup_{\phi\in T} \sup_{t>\frac{|x|}{2}} |\phi_t \star f(x)|. \tag{4.58}$$

Proof. For $|x| \geq 2R$ and $t \leq \frac{|x|}{2}$ we have that $\operatorname{spt} \phi_t(x-\cdot) \subset \mathbb{R}^n \backslash B_R$ and therefore $|\phi_t \star f(x)| = 0$. □

Next we show that we can estimate the L^1-norm of a function by its $\mathcal{H}^1$-norm.

Lemma 4.2.7. *We have that $\mathcal{H}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ and*

$$||f||_{L^1(\mathbb{R}^n)} \leq ||f||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.59}$$

Proof. Let $\phi \in C_c^\infty(B)$ (without loss of generality we assume that $\int_{\mathbb{R}^n} \phi(x)dx = 1$) and for $f \in \mathcal{H}^1(\mathbb{R}^n)$ and every $t > 0$ we define $f_t = \phi_t \star f$. By standard results on convolutions we have that

$$f_t \to f, \tag{4.60}$$

in L^1 as $t \to 0$. Since we also have that $|f_t| \leq f^\star$, for every $t > 0$, and $f^\star \in L^1(\mathbb{R}^n)$ it is easy to see that we have

$$||f||_{L^1(\mathbb{R}^n)} = \lim_{t\to 0}(||f_t||_{L^1(\mathbb{R}^n)} + ||f - f_t||_{L^1(\mathbb{R}^n)}) \leq ||f||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.61}$$

□

In the next Lemma we obtain an important cancellation property for functions in the Hardy space $\mathcal{H}^1$ which also shows that the inclusion in Lemma 4.2.7 is strict.

Lemma 4.2.8. *For every $f \in \mathcal{H}^1(\mathbb{R}^n)$ we have that*

$$\int_{\mathbb{R}^n} f(x)dx = 0. \tag{4.62}$$

Proof. We consider $\theta \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{spt} \theta \subset B$ and $\theta(0) = 1$. Then we have that $||\nabla\theta_t||_{L^\infty} \leq \frac{A}{t^{n+1}}$, where $A = A(\theta)$, and $\operatorname{spt} \theta_t \subset B_t$. Now we let $x \in B_t$ and define $\psi(y) = \frac{t^n}{2A}\theta_t(x - 2ty)$. It is easy to see that $\operatorname{spt} \psi \subset \{y \in \mathbb{R}^n | |x - 2ty| \leq t\}$. But since $2t|y| \leq |x - 2ty| + |x| \leq 2t$ we see that $\operatorname{spt} \psi \subset B$. Moreover we have that $||\nabla\psi||_{L^\infty} \leq \frac{t^{n+1}}{A}||\nabla\theta_t||_{L^\infty} \leq 1$. This implies that $\psi \in T$. Next we calculate

$$\begin{aligned}\psi_{2t} \star f(x) &= \frac{1}{2^n t^n}\int_{\mathbb{R}^n} \psi(\frac{x-y}{2t})f(y)dy \\ &= \frac{1}{2^{n+1}A}\int_{\mathbb{R}^n} \theta_t(y)f(y)dy.\end{aligned}$$

Therefore we conclude that for every $t > 0$ and every $x \in B_t$

$$\begin{aligned}\frac{1}{t^n}|\int_{\mathbb{R}^n} \theta(\frac{y}{t})f(y)dy| &\leq A2^{n+1}|\psi_{2t} \star f(x)| \\ &\leq cf^\star(x).\end{aligned} \tag{4.63}$$

If we now assume that (4.62) is not satisfied we conclude from the dominated convergence theorem that (remember $\theta(0)=1$)

$$\lim_{t\to\infty} |\int_{\mathbb{R}^n} \theta(\frac{y}{t}) f(y)dy| = |\int_{\mathbb{R}^n} f(y)dy| > 0. \tag{4.64}$$

Combining this with (4.63) we get that there exists some $t_0 > 0$ such that

$$f^\star(x)t^n \geq c, \tag{4.65}$$

for every $t > t_0$ and every $x \in B_t$. In particular we can choose $t = |x|+1$ for $t_0 < |x|$ and get

$$f^\star(x) \geq \frac{c}{(|x|+1)^n}, \tag{4.66}$$

which contradicts the assumption $f \in \mathcal{H}^1(\mathbb{R}^n)$. $\square$

In the following Lemma we prove that every L^p-function ($1 < p \leq \infty$) with vanishing mean value is in $\mathcal{H}^1$.

Lemma 4.2.9. *Let $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$, be compactly supported with $\int_{\mathbb{R}^n} f(x)dx = 0$. Then we have $f \in \mathcal{H}^1(\mathbb{R}^n)$.*

Proof. We let $\operatorname{spt} f \subset B_R$ for some $R > 0$. Then we write

$$||f^\star||_{L^1} = \int_{B_{2R}} f^\star(x)dx + \int_{\mathbb{R}^n \setminus B_{2R}} f^\star(x)dx. \tag{4.67}$$

Since $f \in L^p(\mathbb{R}^n)$ with $p > 1$ we see that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and therefore we get from (4.56) that $f^\star(x) \leq cMf(x)$ for every $x \in \mathbb{R}^n$. Using the above considerations, Hölder‘s inequality and Theorem 4.1.4 we conclude for every $p > 1$

$$\begin{aligned}\int_{B_{2R}} f^\star(x)dx \leq& cR^{n-\frac{n}{p}}||f^\star||_{L^p}\\ \leq& cR^{n-\frac{n}{p}}||Mf||_{L^p}\\ \leq& cR^{n-\frac{n}{p}}||f||_{L^p}.\end{aligned} \tag{4.68}$$

Now we calculate for $\phi \in T$ and $x \in \mathbb{R}^n$

$$\begin{aligned}|\phi_t \star f(x)| &= |\int_{B_R} \phi_t(x-y)f(y)dy|\\ &= |\int_{B_R} (\phi_t(x-y) - \phi_t(x))f(y)dy|\\ &\leq ||\nabla\phi_t||_{L^\infty} \int_{B_R} |y||f(y)|dy,\end{aligned}$$

where we used the mean value theorem and the cancellation property $\int_{\mathbb{R}^n} f(y)dy = 0$. Since $||\nabla\phi_t||_{L^\infty} \leq \frac{1}{t^{n+1}}$, for $t > 0$, we continue to estimate for $p > 1$

$$\begin{aligned}|\phi_t \star f(x)| &\leq \frac{R}{t^{n+1}} \int_{B_R} |f(y)|dy\\ &\leq \frac{cR^{n+1-\frac{n}{p}}}{t^{n+1}}||f||_{L^p}.\end{aligned} \tag{4.69}$$

Assuming now that $|x| \geq 2R$ we can apply Lemma 4.2.6 to get

$$\begin{aligned} f^{\star}(x) &= \sup_{\phi \in T} \sup_{t > \frac{|x|}{2}} |\phi_t \star f(x)| \\ &\leq \frac{cR^{n+1-\frac{n}{p}}}{|x|^{n+1}} ||f||_{L^p}. \end{aligned} \tag{4.70}$$

Inserting (4.68) and (4.70) into (4.67) we conclude

$$\begin{aligned} ||f^{\star}||_{L^1} &\leq cR^{n-\frac{n}{p}} ||f||_{L^p} + cR^{n+1-\frac{n}{p}} ||f||_{L^p} \int_{\mathbb{R}^n \setminus B_{2R}} \frac{1}{|x|^{n+1}} dx \\ &\leq cR^{n-\frac{n}{p}} ||f||_{L^p}, \end{aligned} \tag{4.71}$$

which proves the Lemma. □

Remark 4.2.10. *We give an example of a function in $\mathcal{H}^1(\mathbb{R})$ with compact support which does not belong to any L^p, $p > 1$. Let f be defined by*

$$f(x) = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} a_n(x),$$

where

$$a_n(x) = -\frac{n}{2} \chi_{[-\frac{1}{n},0)} + \frac{n}{2} \chi_{[0,\frac{1}{n})}. \tag{4.72}$$

Since $a_n \in L^{\infty}(\mathbb{R})$ and $\int_{\mathbb{R}} a_n(x) dx = 0$ for every $n \geq 2$ we can apply (4.71) to get

$$||a_n||_{\mathcal{H}^1} = ||a_n^{\star}||_{L^1} \leq \frac{c}{n} ||a_n||_{L^{\infty}} \leq c.$$

This implies that

$$||f||_{\mathcal{H}^1} \leq \sum_{n=2}^{\infty} \frac{||a_n||_{\mathcal{H}^1}}{n(\log n)^2} \leq \sum_{n=2}^{\infty} \frac{c}{n(\log n)^2} \leq c.$$

This shows that $f \in \mathcal{H}^1(\mathbb{R})$. Now we estimate for $x \in [0, \frac{1}{2n})$

$$\begin{aligned} f(x) &\geq \sum_{k=n}^{2n} \frac{a_k(x)}{k(\log k)^2} \\ &\geq a_n(x) \int_n^{2n} \frac{dy}{y(\log y)^2} \\ &\geq \frac{n}{2} (\frac{1}{\log n} - \frac{1}{\log 2n}). \end{aligned}$$

This implies for every $p > 1$

$$\begin{aligned} ||f||_{L^p}^p &\geq \int_0^{\frac{1}{2n}} f(x)^p dx \\ &\geq \frac{cn^p}{2(\log n \log 2n)^p n} \\ &\to \infty, \end{aligned}$$

for $n \to \infty$. Hence f does not belong to any L^p for $p > 1$.

Next we prove a compactness result for $\mathcal{H}^1(\mathbb{R}^n)$ functions (note that the corresponding result is not true in L^1).

Lemma 4.2.11. *Let $f_k \in \mathcal{H}^1(\mathbb{R}^n)$, $k \in \mathbb{N}$, with $||f_k||_{\mathcal{H}^1(\mathbb{R}^n)} \leq c$. Then there exists a subsequence f_{k_j}, $j \in \mathbb{N}$, such that $f_{k_j} \to f \in \mathcal{H}^1(\mathbb{R}^n)$ in the sense of distributions and moreover*

$$||f||_{\mathcal{H}^1(\mathbb{R}^n)} \leq \operatorname{liminf}_{j\to\infty} ||f_{k_j}||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.73}$$

Proof. Because of Lemma 4.2.7 we see that also the L^1-norm of f_k is uniformly bounded. Therefore we get the existence of a subsequence f_{k_j} which converges weakly to some $f \in \mathcal{D}'(\mathbb{R}^n)$ in the sense of distributions. From this it follows that for every $\phi \in T$ and every $t > 0$ we have

$$|\phi_t \star f(x)| = \lim_{j\to\infty} |\phi_t \star f_{k_j}(x)| \leq \operatorname{liminf}_{j\to\infty} f^\star_{k_j}(x). \tag{4.74}$$

Taking the supremum over all ϕ and t this yields

$$0 \leq f^\star(x) \leq \operatorname{liminf}_{j\to\infty} f^\star_{k_j}(x). \tag{4.75}$$

By Fatou's Lemma this gives

$$\begin{aligned} ||f||_{\mathcal{H}^1} = ||f^\star||_{L^1} &\leq \int_{\mathbb{R}^n} \operatorname{liminf}_{j\to\infty} f^\star_{k_j}(x)dx \\ &\leq \operatorname{liminf}_{j\to\infty} \int_{\mathbb{R}^n} f^\star_{k_j}(x)dx = \operatorname{liminf}_{j\to\infty} ||f_{k_j}||_{\mathcal{H}^1}. \end{aligned} \tag{4.76}$$

□

In the next Theorem (which was proved by Coifman, Lions, Meyer & Semmes [7]) we give a very important example of a function belonging to $\mathcal{H}^1$.

Theorem 4.2.12. *Let $E \in L^p(\mathbb{R}^n, \mathbb{R}^n)$ and $B \in W^{1,q}(\mathbb{R}^n)$, with $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\operatorname{div} E = 0$ weakly. Then we have that $E \cdot \nabla B \in \mathcal{H}^1(\mathbb{R}^n)$ and*

$$||E \cdot \nabla B||_{\mathcal{H}^1(\mathbb{R}^n)} \leq c||E||_{L^p(\mathbb{R}^n)}||\nabla B||_{L^q(\mathbb{R}^n)}. \tag{4.77}$$

Proof. Let $\phi \in T$. For each $x \in \mathbb{R}^n$ and $t > 0$ we let $B^x_t = \frac{1}{|B_t(x)|}\int_{B_t(x)} B(x)dx$. Then we calculate, using $\operatorname{div} E = 0$ weakly

$$\begin{aligned} |\phi_t \star (E \cdot \nabla B)(x)| &= |t^{-n} \int_{B_t(x)} \phi(\frac{x-y}{t})(E \cdot \nabla(B - B^x_t))(y)dy| \\ &= |t^{-(n+1)} \int_{B_t(x)} \nabla\phi(\frac{x-y}{t})(E(B - B^x_t))(y)dy| \\ &\leq \frac{||\nabla\phi||_{L^\infty(\mathbb{R}^n)}}{t}(\frac{1}{|B_t(x)|}\int_{B_t(x)} E^\alpha)^{\frac{1}{\alpha}}(\frac{1}{|B_t(x)|}\int_{B_t(x)} |B - B^x_t|^{\alpha'})^{\frac{1}{\alpha'}}, \end{aligned} \tag{4.78}$$

where $1 < \alpha < p$ and $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Using the Poincare-Sobolev inequality and the fact that $||\nabla\phi||_{L^\infty} \leq 1$, we get

$$|\phi_t \star (E \cdot \nabla B)(x)| \leq c(\frac{1}{|B_t(x)|}\int_{B_t(x)} E^\alpha)^{\frac{1}{\alpha}}(\frac{1}{|B_t(x)|}\int_{B_t(x)} |\nabla B|^\beta)^{\frac{1}{\beta}}, \tag{4.79}$$

where $1 < \beta < q$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1 + \frac{1}{n}$. Taking the supremum over all $\phi \in T$ and all $t > 0$ we conclude

$$(E \cdot \nabla B)^\star(x) \le c(M(|E|^\alpha)(x))^{\frac{1}{\alpha}}(M(|\nabla B|^\beta)(x))^{\frac{1}{\beta}}. \tag{4.80}$$

Using Hölder‘s inequality and Theorem 4.1.4 we estimate

$$\begin{aligned}||E \cdot \nabla B||_{\mathcal{H}^1(\mathbb{R}^n)} =&||(E \cdot \nabla B)^\star||_{L^1(\mathbb{R}^n)}\\ \le& c||M(|E|^\alpha)||^{\frac{1}{\alpha}}_{L^{\frac{p}{\alpha}}(\mathbb{R}^n)}||M(|\nabla B|^\beta)||^{\frac{1}{\beta}}_{L^{\frac{q}{\beta}}(\mathbb{R}^n)}\\ \le& c||E||_{L^p(\mathbb{R}^n)}||\nabla B||_{L^q(\mathbb{R}^n)}.\end{aligned} \tag{4.81}$$

This proves the Theorem. □

As a Corollary we obtain that the nonlinearities which occured in the Euler-Lagrange equations of constant mean curvature surfaces and harmonic maps into spheres also belong to $\mathcal{H}^1$.

Corollary 4.2.13. *Let $u, v \in W^{1,2}(\mathbb{R}^2)$, then $\nabla u \nabla^\perp v \in \mathcal{H}^1(\mathbb{R}^2)$ and*

$$||\nabla u \nabla^\perp v||_{\mathcal{H}^1(\mathbb{R}^2)} \le c||\nabla u||_{L^2(\mathbb{R}^2)}||\nabla v||_{L^2(\mathbb{R}^2)}. \tag{4.82}$$

Moreover, if $w \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$, then $\det(\nabla w) \in \mathcal{H}^1(\mathbb{R}^2)$ and

$$||\det(\nabla w)||_{\mathcal{H}^1(\mathbb{R}^2)} \le c||\nabla w||^2_{L^2(\mathbb{R}^2)}. \tag{4.83}$$

Proof. Since $\operatorname{div} \nabla^\perp v = 0$ weakly we can apply Theorem 4.2.12 with $B = u$ and $E = \nabla^\perp v$ to get the estimate (4.82). If we now also note that $\det(\nabla w) = \nabla w^1 \nabla^\perp w^2$ we see that the estimate (4.83) follows from the estimate (4.82). □

Remark 4.2.14. *More generally we have for $w \in W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ that $\det(\nabla w) \in \mathcal{H}^1(\mathbb{R}^n)$. (Hint: Use $(\det \nabla w) id = \nabla w (\operatorname{adj} \nabla w)$ and Lemma 4.1.9.)*
Altogether this yields an improvement of Theorem 4.1.11.

4.3 Atomic decomposition

In this section we prove that every element in $\mathcal{H}^1$ can be decomposed into so called atoms.

Definition 4.3.1. *A function $a \in L^1(\mathbb{R}^n)$ with the properties*

$$\operatorname{spt} a \subset B, \tag{4.84}$$

$$||a||_{L^\infty(\mathbb{R}^n)} \le \frac{1}{|B|}, \tag{4.85}$$

$$\int_{\mathbb{R}^n} a(x)dx = 0, \tag{4.86}$$

where $B \subset \mathbb{R}^n$ is a ball, is called $\mathcal{H}^1$-atom.

An example of $\mathcal{H}^1$-atoms is given by the functions in (4.72).

Lemma 4.3.2. *Let $a \in L^1(\mathbb{R}^n)$ be an $\mathcal{H}^1$-atom. Then we have that*

$$||a||_{\mathcal{H}^1(\mathbb{R}^n)} \le c, \tag{4.87}$$

where the constant c is independent of the atom.

Proof. From the definition of an $\mathcal{H}^1$-atom we can assume that $\operatorname{spt} a \subset B = B_R$. Since a is additionally in $L^\infty(\mathbb{R}^n)$ with vanishing mean value we can apply Lemma 4.2.9 (more precisely estimate (4.71)) to get

$$||a||_{\mathcal{H}^1(\mathbb{R}^n)} \le cR^n ||a||_{L^\infty(\mathbb{R}^n)} \le c. \tag{4.88}$$

This proves the Lemma. □

In the next Lemma we show that a sum of $\mathcal{H}^1$-atoms also belongs to $\mathcal{H}^1$.

Lemma 4.3.3. *Let $a_k \in L^1(\mathbb{R}^n)$, $k \in \mathbb{N}$, be a sequence of $\mathcal{H}^1$-atoms and let λ_k, $k \in \mathbb{N}$, be sequence in $\mathbb{R}$ with $\sum_{k=1}^\infty |\lambda_k| < \infty$. Then we have that $f = \sum_{k=1}^\infty \lambda_k a_k \in \mathcal{H}^1(\mathbb{R}^n)$ with*

$$||f||_{\mathcal{H}^1(\mathbb{R}^n)} \le c \sum_{k=1}^\infty |\lambda_k|. \tag{4.89}$$

Proof. For $N \in \mathbb{N}$ we consider the bounded function

$$f_N(x) = \sum_{k=1}^N \lambda_k a_k(x).$$

Using Lemma 4.3.2 and the assumption for $\sum |\lambda_k|$ we conclude that $f_N \in \mathcal{H}^1(\mathbb{R}^n)$ and moreover with the help of estimate (4.87) we get

$$\begin{aligned} ||f_N - f_M||_{\mathcal{H}^1} &\le \sum_{k=M}^N |\lambda_k| ||a_k||_{\mathcal{H}^1} \\ &\le c \sum_{k=M}^N |\lambda_k|, \end{aligned} \tag{4.90}$$

for every $N \ge M \in \mathbb{N}$. If we now define

$$f = f_1 + \sum_{N=1}^\infty (f_{N+1} - f_N) \in L^1(\mathbb{R}^n),$$

we see that (using (4.90))

$$\begin{aligned} ||f||_{\mathcal{H}^1} \le& c|\lambda_1| + \sum_{N=1}^\infty ||f_{N+1} - f_N||_{\mathcal{H}^1} \\ \le& c|\lambda_1| + \sum_{N=1}^\infty |\lambda_N| \\ \le& c \sum_{N=1}^\infty |\lambda_N|, \end{aligned} \tag{4.91}$$

which shows that $f \in \mathcal{H}^1(\mathbb{R}^n)$. Moreover we have (again using (4.90))

$$\begin{aligned}||f - f_N||_{\mathcal{H}^1} \le& \sum_{k=N}^{\infty} ||f_{k+1} - f_k||_{\mathcal{H}^1} \\ \le& c \sum_{k=N}^{\infty} |\lambda_k|. \end{aligned} \tag{4.92}$$

This shows that $f_N \to f$ in $\mathcal{H}^1(\mathbb{R}^n)$ and proves the Lemma. □

The goal of the rest of this section is to prove a statement converse to the one in Lemma 4.3.3, namely that every function $f \in \mathcal{H}^1(\mathbb{R}^n)$ can be decomposed into a sum of $\mathcal{H}^1$-atoms. In order to do this we need a version of the Whitney decomposition which can be found for example in [32].

Lemma 4.3.4. *Let $F \subset \mathbb{R}^n$ be non-empty and closed and let $\Omega = \mathbb{R}^n \backslash F$. Then there exists a collection of cubes $\mathcal{F} = \{Q_1, \dots, Q_k, \dots\}$ so that*

(i) $\cup_k Q_k = \Omega$,

(ii) the Q_k are mutually disjoint and

(iii) $\operatorname{diam} Q_k \le d(Q_k, F) \le 4 \operatorname{diam} Q_k$ *for every* $k \in \mathbb{N}$.

Moreover there exists $1 < b$, such that if $Q_k^\star$ denotes the cube which has the same center as Q_k but whose side length is $(1 + b)$ times the side length of Q_k, then these cubes satisfy $\cup_k Q_k = \cup_k Q_k^\star$ and the cubes $\{Q_k^\star\}$ have the bounded intersection property.

Proof. We let $\mathcal{Q}_0$ be the set of all cubes with side length 1 and whose corners are in $\mathbb{Z}^n$. Then we let $\mathcal{Q}_l = 2^{-l}\mathcal{Q}_0$ be the cubes with side length 2^{-l} and whose cornes are in $(2^{-l}\mathbb{Z})^n$. The diameter of the cubes in $\mathcal{Q}_l$ is $\sqrt{n}2^{-l}$. Now we let $\Omega = \cup_{l=-\infty}^{\infty} \Omega_l$ where $\Omega_l = \{x \in \Omega | \sqrt{n}2^{-l+1} < d(x, F) \le \sqrt{n}2^{-l+2}\}$ and we define

$$\mathcal{F}_0 = \cup_{l \in \mathbb{Z}} \{Q \in \mathcal{Q}_l | Q \cap \Omega_l \neq \emptyset\}.$$

For $Q \in \mathcal{F}_0$, e.g. $Q \in \mathcal{Q}_l$ and $x \in Q \cap \Omega_l$, we have

$$d(Q, F) \le d(x, F) \le \sqrt{n}2^{-l+2} = 4 \operatorname{diam} Q \tag{4.93}$$

and

$$d(Q, F) \ge d(x, F) - \operatorname{diam} Q \ge \sqrt{n}2^{-l+1} - \sqrt{n}2^{-l} \ge \operatorname{diam} Q. \tag{4.94}$$

This proves (iii) of the Lemma for all cubes in $Q \in \mathcal{F}_0$ and shows in particular that $Q \subset \Omega$. From the definition we therefore get

$$\Omega = \cup_{Q \in \mathcal{F}_0} Q. \tag{4.95}$$

Next we let $\mathcal{F}$ be the family of the maximal cubes of $\mathcal{F}_0$, i.e. all cubes $Q \in \mathcal{F}_0$ such that if $Q' \in \mathcal{F}_0$, $Q \subset Q' \Rightarrow Q = Q'$. Since $F \neq \emptyset$ we see that every $x \in \Omega$ lies in at least one cube with maximal side length. This implies that $\mathcal{F}$ still covers Ω. The fact that the cubes in $\mathcal{F}$ are mutually disjoint follows directly from the definition.
It remains to prove the last statement of the Lemma. We prove this result in three steps:

1) If two cubes $Q_1, Q_2 \in \mathcal{F}$ touch each other, then we have

$$\frac{1}{4} \operatorname{diam} Q_2 \leq \operatorname{diam} Q_1 \leq 4 \operatorname{diam} Q_2. \tag{4.96}$$

To see this we note that since $Q_1 \in \mathcal{F}$ we have $d(Q_1, F) \leq 4 \operatorname{diam} Q_1$. This implies $d(Q_2, F) \leq 4 \operatorname{diam} Q_1 + \operatorname{diam} Q_1 = 5 \operatorname{diam} Q_1$, since Q_1 and Q_2 touch. Since $Q_2 \in \mathcal{F}$ we conclude $\operatorname{diam} Q_2 \leq d(Q_2, F) \leq 5 \operatorname{diam} Q_1$. By construction we have $\operatorname{diam} Q_2 = 2^k \operatorname{diam} Q_1$, for some $k \in \mathbb{Z}$, and therefore $\operatorname{diam} Q_2 \leq 4 \operatorname{diam} Q_1$. The other inequality follows from symmetry considerations.

2) Let $N = (12)^n$. If $Q \in \mathcal{F}$ then there are at most N cubes in $\mathcal{F}$ which touch Q.
If $Q \in \mathcal{Q}_l$ it is easy to see that there are 3^n cubes (including Q) which belong to $\mathcal{Q}_l$ and touch Q. Next, each cube in $\mathcal{Q}_l$ can contain at most 4^n cubes in $\mathcal{F}$ which have diameter larger than or equal to $\frac{1}{4} \operatorname{diam} Q$. Combining this with (4.96) finishes the proof of statement 2).

3) Let $0 < b < \frac{1}{4}$. Then each point $x \in \Omega$ is contained in at most N of the cubes $Q_k^\star$, $Q_k \in \mathcal{F}$.
Let $Q, Q_k \in \mathcal{F}$. We claim that $Q_k^\star$ intersects Q only if Q_k touches Q. In fact, if we consider the union of Q_k with all cubes in $\mathcal{F}$ which touch Q_k it is clear that this union contains $Q_k^\star$ (since the diameter of the cubes touching Q_k are all larger than or equal to $\frac{1}{4} \operatorname{diam} Q_k$). Therefore Q intersects $Q_k^\star$ only if Q touches Q_k. Since any point $x \in \Omega$ is contained in a cube Q we see from step 2) that there are at most N cubes $Q_k^\star$ which contain x.

Since we also have that if $Q_k \in \mathcal{F}$ then $Q_k^\star \subset \Omega$ we get $\Omega = \cup_{Q \in \mathcal{F}} Q^\star$ and this finishes the proof of the Lemma. □

The key step in the atomic decomposition is the following variant of the Calderon-Zygmund decomposition.

Theorem 4.3.5. *Let $f \in \mathcal{H}^1(\mathbb{R}^n)$ and let $\alpha > 0$. Then there exists a decomposition $f = g + b$, with $b = \sum_{k=1}^\infty b_k$, and a countable family of cubes C_k, $k \in \mathbb{N}$, such that*

(i) For a.e. $x \in \mathbb{R}^n$ we have

$$|g(x)| \leq c\alpha. \tag{4.97}$$

(ii) Every function b_k is supported in C_k, satisfies $\int_{C_k} b_k(x)dx = 0$ and

$$||b_k||_{\mathcal{H}^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} b_k^\star(x)dx \leq c \int_{C_k} f^\star(x)dx. \tag{4.98}$$

(iii) The countable family of cubes C_k has the bounded intersection property and if we set $\Omega = \cup_{k=1}^\infty C_k$ we have

$$\Omega = \{x \in \mathbb{R}^n | f^\star(x) > \alpha\}. \tag{4.99}$$

Proof. For $\alpha > 0$ and $f \in \mathcal{H}^1(\mathbb{R}^n)$ we set

$$\Omega = \{x \in \mathbb{R}^n | f^\star(x) > \alpha\}$$

and note that Ω is open. (This follows from the fact that $f^\star$ is obtained by taking the sup over smooth functions and is therefore lower-semicontinuous. Hence the set $\mathbb{R}^n \backslash \Omega = \{x \in \mathbb{R}^n | f^\star(x) \leq \alpha\}$ is compact.) Next we apply the Whitney decomposition of Lemma 4.3.4 to Ω and $F = \mathbb{R}^n \backslash \Omega$ and we denote the cubes which we obtain from this decomposition by C_k'. Moreover we can choose two constants $1 < a < b \in \mathbb{R}$ such that, if $\tilde{C}_k$ and C_k denote the cubes having the same center as C_k' but scaled with the factors a and b, then $\Omega = \cup_{k=1}^\infty C_k$ and the family C_k has the bounded intersection property.
Next we consider a positive function ξ satisfying

$$\xi \in C_c^\infty([-\frac{a}{2}, \frac{a}{2}]^n), \quad \xi = 1 \quad \text{on} \quad [-\frac{1}{2}, \frac{1}{2}]^n.$$

Then we set

$$\xi_k = \xi(\frac{x - c_k}{l_k}), \tag{4.100}$$

where c_k denotes the center of the cube C_k' and l_k the length of its edges. Note that $\xi_k \in C_c^\infty(\tilde{C}_k)$ and $\xi_k \equiv 1$ on C_k'. If we define

$$\eta_k(x) = \frac{\xi_k(x)}{\sum_{j=1}^\infty \xi_j(x)} \in C_c^\infty(\tilde{C}_k) \tag{4.101}$$

it is not difficult to see that $\{\eta_k\}$ forms a partition of unity subordinated to $\tilde{C}_k$ (here we use the finite intersection property of the $\tilde{C}_k$'s in order to ensure that the sum in the denominator is finite). Moreover we observe that

$$l_k^n = |C_k'| \leq \int_{C_k} \eta_k(x) dx \leq |C_k| = b^n l_k^n \tag{4.102}$$

and

$$||\nabla \eta_k||_{L^\infty} \leq \frac{c}{l_k}, \tag{4.103}$$

for every $k \in \mathbb{N}$, where c does not depend on k.

Now for f and η_k as above we define

$$b_k(x) = (f(x) - a_k)\eta_k(x), \tag{4.104}$$

where

$$a_k = \frac{\int_{C_k} \eta_k(x) f(x) dx}{\int_{C_k} \eta_k(x) dx}.$$

Therefore b_k is supported in C_k and we see that

$$\int_{C_k} b_k(x) dx = 0. \tag{4.105}$$

Moreover we define

$$g(x) = \chi_{\mathbb{R}^n \setminus \Omega}(x) f(x) + \chi_\Omega(x) \sum_{k=1}^{\infty} a_k \eta_k(x). \tag{4.106}$$

Next we claim that

$$|a_k| \leq c. \tag{4.107}$$

In order to see this we note that

$$a_k = \int_{C_k} f(x)\phi_k(x)dx = (f \star \phi)(0),$$

where $\phi = \frac{\eta_k}{\int_{C_k} \eta_k(x)dx}$ and therefore

$$||\nabla \phi||_{L^\infty} \leq \frac{||\nabla \eta_k||_{L^\infty}}{\int_{C_k} \eta_k(x)dx} \leq c l_k^{-n-1},$$

where we used (4.102) and (4.103) in the last step. Therefore we can argue as in the proof of Lemma 4.2.8 (more precisely as in the proof of (4.63) with $z \in B_{Cl_k}(c_k) \cap \Omega^c$, the constant C can be chosen in such a way because we consider a Whitney decomposition) to get

$$|a_k| = |(f \star \phi_k)(0)| \leq c f^\star(z) \leq c\alpha. \tag{4.108}$$

Moreover we see that

$$|a_k| \leq c f^\star(x), \tag{4.109}$$

for every $x \in C_k$. Now we want to show that the statement (i) of the Theorem holds. In the case $x \notin \Omega$ we have

$$|g(x)| = |f(x)| \leq f^\star(x) \leq \alpha, \tag{4.110}$$

where we used the definition of Ω. If $x \in \Omega$ we have, using (4.108)

$$|g(x)| \leq \sum_{k=1}^{\infty} |a_k| \eta_k \leq c\alpha. \tag{4.111}$$

Now it remains to prove statement (ii) of the Theorem.
We claim that if we have

$$b_k^\star(x) \leq c f^\star(x), \quad \text{for} \quad x \in C_k \tag{4.112}$$

$$b_k^\star(x) \leq \frac{c\alpha l_k^{n+1}}{|x - c_k|^{n+1}} \quad \text{for} \quad x \notin C_k, \tag{4.113}$$

then we are done. To see this we estimate

$$\begin{aligned} \int_{\mathbb{R}^n} b_k^\star(x)dx &= \int_{C_k} b_k^\star(x)dx + \int_{\mathbb{R}^n \setminus C_k} b_k^\star(x)dx \\ &\leq c \int_{C_k} f^\star(x)dx + \int_{\mathbb{R}^n \setminus C_k} \frac{c\alpha l_k^{n+1}}{|x - c_k|^{n+1}} dx. \end{aligned}$$

The last integral can be estimated by (remember $\alpha < f^\star(x)$ for $x \in C_k$)

$$\begin{aligned}\int_{\mathbb{R}^n \setminus C_k} \frac{c\alpha l_k^{n+1}}{|x-c_k|^{n+1}} dx &\le c\alpha l_k^{n+1} \int_{\mathbb{R}^n \setminus B_{\frac{l_k}{2}}(c_k)} \frac{1}{|x-c_k|^{n+1}} dx \\ &\le c\alpha l_k^n \\ &\le c\alpha |C_k| \\ &\le c \int_{C_k} f^\star(x) dx.\end{aligned}$$

Altogether this shows that

$$\int_{\mathbb{R}^n} b_k^\star(x) dx \le c \int_{C_k} f^\star(x) dx. \tag{4.114}$$

We are left with proving the estimates (4.112) and (4.113). First we write for $\phi \in T$ and $x \in C_k$

$$\int_{\mathbb{R}^n} \phi_t(x-y) f(y) \eta_k(y) dy = \int_{\mathbb{R}^n} \varphi_{t,\phi}(y) f(y) dy,$$

where $\varphi_{t,\phi}(y) = \eta_k(y)\phi_t(x-y)$. We estimate, using (4.103)

$$||\nabla \varphi_{t,\phi}||_{L^\infty} \le \frac{c||\nabla \eta_k||_{L^\infty}}{t^n} + c||\nabla \phi_t||_{L^\infty} \tag{4.115}$$

$$\le c\left(\frac{1}{l_k t^n} + \frac{1}{t^{n+1}}\right). \tag{4.116}$$

Now we have two cases. The first one is that $t \le l_k$. In this situation we have that $||\nabla \varphi_{t,\phi}||_{L^\infty} \le \frac{c}{t^{n+1}}$ and we can again argue as in the proof of (4.63) ($\varphi_{t,\phi} \in C_c^\infty(B_t(x))$) to get

$$\sup_{\phi \in T} \sup_{0<t\le l_k} |\int_{\mathbb{R}^n} \varphi_{t,\phi}(y) f(y) dy| \le c f^\star(x). \tag{4.117}$$

In the case $t > l_k$ we have that $||\nabla \varphi_{t,\phi}||_{L^\infty} \le \frac{c}{l_k^{n+1}}$ and $\varphi_{t,\phi}$ is supported in $B_{cl_k(x)}$ (remember $\eta_k \in C_c^\infty(C_k)$), where c is chosen such that $C_k \subset B_{cl_k}(x)$. Therefore we can argue as above to get

$$\sup_{\phi \in T} \sup_{t>l_k} |\int_{\mathbb{R}^n} \varphi_{t,\phi}(y) f(y) dy| \le c f^\star(x). \tag{4.118}$$

Combining (4.117) and (4.118) we arrive at

$$(f\eta_k)^\star(x) = \sup_{\phi \in T} \sup_{0<t} |\int_{\mathbb{R}^n} \phi_t(x-y) f(y) \eta_k(y) dy| \tag{4.119}$$

$$\le c f^\star(x). \tag{4.120}$$

This implies that for $x \in C_k$ we have

$$b_k^\star(x) \le (f\eta_k)^\star(x) + |a_k| \eta_k^\star(x) \le c f^\star(x), \tag{4.121}$$

where we used (4.109) to get that $|a_k|\eta_k^\star(x) \le |a_k| \le cf^\star(x)$. This shows (4.112).
We use (4.105) to get

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_t(x-y)b_k(y)dy =& \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-c_k))b_k(y)dy \\ =& I_1 - I_2, \end{aligned} \tag{4.122}$$

where

$$\begin{aligned} I_1(x) =& \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-c_k))f(y)\eta_k(y)dy \\ I_2(x) =& \int_{\mathbb{R}^n} (\phi_t(x-y) - \phi_t(x-c_k))a_k(y)\eta_k(y)dy. \end{aligned}$$

This time we define $\psi_{t,\phi}(y) = (\phi_t(x-y) - \phi_t(x-c_k))\eta_k(y)$. Now we note that $\int_{\mathbb{R}^n} \phi_t(x-y)b_k(y)dy \neq 0$ only if the supports of b_k and $\phi_t(x-\cdot)$ intersect. This is the case if $t \ge c|x-c_k|$. From this we get by the mean value theorem

$$\begin{aligned} |\psi_{t,\phi}(y)| \le& |\eta_k(y)| ||\nabla\phi||_{L^\infty}|y-c_k| \\ \le& \frac{cl_k}{|x-c_k|^{n+1}}. \end{aligned} \tag{4.123}$$

Another application of the mean value theorem and (4.103) yields

$$\begin{aligned} |\nabla\psi_{t,\phi}(y)| \le& |\nabla\phi_t(x-y)| + |\nabla\eta_k(y)||\phi_t(x-y) - \phi_t(x-c_k)| \\ \le& \frac{c}{t^{n+1}} + \frac{c}{l_k}\frac{cl_k}{t^{n+1}} \\ \le& \frac{c}{t^{n+1}}. \end{aligned} \tag{4.124}$$

Next we argue again as in the proof of (4.63) (this time applied to the function $\psi_{t,\phi} \in C_c^\infty(B_{cl_k}(c_k))$, where c is chosen such that there exists $z \in B_{cl_k}(c_k) \cap \Omega^c$) to get

$$|\psi_{t,\phi} \star f| = |I_1(x)| \le \frac{cl_k^{n+1} f^\star(z)}{|x-c_k|^{n+1}} \le \frac{c\alpha l_k^{n+1}}{|x-c_k|^{n+1}}, \tag{4.125}$$

where we also used that by (4.124) we have $|\nabla\psi_{t,\phi}(y)| \le \frac{A}{l_k^{n+1}}$ with $A = \frac{cl_k^{n+1}}{|x-c_k|^{n+1}}$. On the other hand I_2 can be estimated by

$$|I_2(x)| \le c|a_k||C_k|||\psi_{t,\phi}||_{L^\infty} \le c\frac{c\alpha l_k^{n+1}}{|x-c_k|^{n+1}}, \tag{4.126}$$

where we used (4.123) and (4.108). Combining (4.122), (4.125) and (4.126) we prove (4.113) and therefore the Theorem. □

In the next Theorem we finally prove the atomic decomposition.

Theorem 4.3.6. *Let $f \in \mathcal{H}^1(\mathbb{R}^n)$. Then there exists a sequence a_k, $k \in \mathbb{N}$, of $\mathcal{H}^1$-atoms and a sequence $\lambda_k \in \mathbb{R}$, such that*

$$f = \sum_{k=1}^\infty \lambda_k a_k, \tag{4.127}$$

where the convergence is in the $\mathcal{H}^1$-norm, and moreover

$$\sum_{k=1}^{\infty} |\lambda_k| \leq c||f||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.128}$$

Proof. For every $j \in \mathbb{Z}$ we apply Theorem 4.3.5 to $f \in \mathcal{H}^1(\mathbb{R}^n)$ and $\alpha = 2^j > 0$. From this we get the decomposition

$$f(x) = g^j(x) + b^j(x) = g^j(x) + \sum_{k=1}^{\infty} b_k^j, \tag{4.129}$$

and the countable family of cubes $\{C_k^j\}_{k\in\mathbb{N}}$. We estimate

$$\begin{aligned} ||f - g^j||_{\mathcal{H}^1} = ||b^j||_{\mathcal{H}^1} &\leq \sum_k ||b_k^j||_{\mathcal{H}^1} \\ &\leq c \sum_k \int_{C_k^j} f^\star(x)dx \\ &\leq c \int_{\Omega^j} f^\star(x)dx \\ &= c \int_{\{x\in\mathbb{R}^n | f^\star(x) > 2^j\}} f^\star(x)dx, \end{aligned} \tag{4.130}$$

where we used (4.98) in the second line, the bounded intersection property of $\Omega^j = \cup_k C_k^j$ in the third line and (4.99) in the last step. Since $f^\star \in L^1(\mathbb{R}^n)$ we therefore conclude that

$$||f - g^j||_{\mathcal{H}^1} \to 0, \tag{4.131}$$

as $j \to \infty$. Since by (4.97) we also know that $|g^j(x)| \leq c2^j$ a.e. we conclude that

$$g^j \to 0, \tag{4.132}$$

in the sense of distributions as $j \to -\infty$. Combining (4.131) and (4.132) we get

$$\begin{aligned} f &= \lim_{N\to\infty} g^N + \lim_{N\to-\infty} g^N \\ &= \lim_{N\to\infty} \sum_{j=1}^{N} (g^{j+1} - g^j) + \lim_{N\to\infty} (\sum_{j=0}^{-N} (g^{j+1} - g^j) \\ &= \sum_{j=-\infty}^{\infty} (g^{j+1} - g^j), \end{aligned} \tag{4.133}$$

where the convergence is in the sense of distributions. From (4.129) we have that

$$g^{j+1}(x) - g^j(x) = (f(x) - b^{j+1}(x)) - (f(x) - b^j(x)) = b^j(x) - b^{j+1}(x). \tag{4.134}$$

Therefore $g^{j+1} - g^j$ is supported in $\Omega^j \supset \Omega^{j+1}$. Hence we can write

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} (g^{j+1} - g^j)\eta_k^j, \tag{4.135}$$

where $\eta_k^j \in C_c^\infty(C_k^j)$ is the partition of unity subordinated to $\{C_k^j\}$ which was defined in the proof of Theorem 4.3.5. From (4.97) we also get for a.e. $x \in \mathbb{R}^n$

$$|g^{j+1}(x) - g^j(x)| \leq C2^j. \tag{4.136}$$

Now we define real numbers

$$\lambda_{j,k} = C2^j |B_k^j|, \tag{4.137}$$

where C is the constant from (4.136) and B_k^j is the smallest ball containing C_k^j, and functions

$$a_{j,k}(x) = \frac{1}{\lambda_{j,k}} A_{j,k}, \tag{4.138}$$

where $A_{j,k}$ is supported in C_k^j and is defined by

$$A_{j,k} = (g^{j+1}(x) - g^j(x))\eta_k^j(x). \tag{4.139}$$

This implies in particular that $a_{j,k}$ is supported in B_k^j and because of (4.136) and (4.137) we have

$$||a_{j,k}||_{L^\infty} \leq \frac{1}{\lambda_{j,k}} ||g^{j+1} - g^j||_{L^\infty} \leq \frac{1}{|B_k^j|}. \tag{4.140}$$

Moreover we have that

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} |\lambda_{j,k}| &\leq C \sum_{j=-\infty}^{\infty} 2^j |\Omega^j| \\
&= c \sum_{j=-\infty}^{\infty} 2^j |\{x \in \mathbb{R}^n | f^\star(x) > 2^j\}| \\
&= c \sum_{j=-\infty}^{\infty} (\sum_{k=-\infty}^{j} 2^k) |\{x \in \mathbb{R}^n | 2^{j+1} \geq f^\star(x) > 2^j\}| \\
&\leq c \sum_{j=-\infty}^{\infty} 2^j |\{x \in \mathbb{R}^n | 2^{j+1} \geq f^\star(x) > 2^j\}| \\
&\leq c \int_{\mathbb{R}^n} f^\star(x) dx \\
&= c||f||_{\mathcal{H}^1}.
\end{aligned} \tag{4.141}$$

Because of (4.135), (4.138) and (4.139) we also get that

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \lambda_{j,k} a_{j,k}. \tag{4.142}$$

This would yield the desired atomic decomposition of f if we would additionally have that $\int_{\mathbb{R}^n} a_{j,k}(x) dx = 0$. Since this is not true we have to modify the decomposition.
First we note that because of (4.134) and (4.139) we can write

$$A_{j,k}(x) = (b^j(x) - b^{j+1}(x))\eta_k^j(x) = b_k^j(x) - \sum_{l=1}^{\infty} b_l^{j+1}(x)\eta_k^j(x). \tag{4.143}$$

Then we define new functions

$$B_{j,k}(x) = A_{j,k}(x) + \sum_{l=1}^{\infty} b_{l,k} \eta_l^{j+1}(x), \tag{4.144}$$

where

$$b_{l,k} = \frac{\int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^j(x)dx}{\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx}. \tag{4.145}$$

This implies that

$$\begin{aligned}\int_{\mathbb{R}^n} B_{j,k}(x)dx =& \int_{\mathbb{R}^n} A_{j,k}(x)dx + \sum_{l=1}^{\infty} \int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^j(x)dx \\ =& \int_{\mathbb{R}^n} b_k^j(x)dx - \sum_{l=1}^{\infty} \int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^j(x)dx + \sum_{l=1}^{\infty} \int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^j(x)dx \\ =& 0. \end{aligned} \tag{4.146}$$

Moreover we have that

$$\sum_{k=1}^{\infty} b_{l,k} = \frac{\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} b_l^{j+1}(x)\eta_k^l(x)dx}{\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx} = \frac{\int_{\mathbb{R}^n} b_l^{j+1}(x)dx}{\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx} = 0. \tag{4.147}$$

and therefore

$$\sum_{k=1}^{\infty} B_{j,k} = \sum_{k=1}^{\infty} A_{j,k}. \tag{4.148}$$

This shows that (using (4.135) and (4.139))

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} B_{j,k}. \tag{4.149}$$

Now we have to study the numbers $b_{l,k}$ in more detail. Since $\operatorname{spt} b_l^{j+1} \subset C_l^{j+1}$ we see that $b_{l,k} \neq 0$ only if $C_k^j \cap C_l^{j+1} \neq \emptyset$. This shows that $\operatorname{spt} B_{j,k} \subset B_k^j$. Next we want to show that $|B_{j,k}(x)| \leq c2^j$. To show this we need to estimate $|b_{l,k}|$. First of all we note that if $C_k^j \cap C_l^{j+1} \neq \emptyset$ we have

$$c_1 \operatorname{diam} C_k^j \leq \operatorname{diam} C_l^{j+1} \leq c_2 \operatorname{diam} C_k^j.$$

To see this we use the properties of the Whitney decomposition (Lemma 4.3.4) to conclude

$$\operatorname{diam} C_l^{j+1} \leq d(C_l^{j+1}, (\Omega^{j+1})^c) \leq d(C_l^{j+1}, (\Omega^j)^c) \leq \operatorname{diam} C_k^j + d(C_k^j, (\Omega^j)^c) \leq c \operatorname{diam} C_k^j.$$

The other inequality follows from symmetry considerations. Hence we also have that

$$cl_k^j \leq l_l^{j+1} \leq cl_k^j,$$

where l_k^j denotes the side length of the cube C_k^j. Therefore if we define

$$\phi(x) = \frac{\eta_k^j(x)}{\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx}$$

we have that $\operatorname{spt}\phi \subset B_k^j \subset B_{C_1 l_l^{j+1}}(c_l^{j+1})$, where C_1 is chosen such that $B_{C_1 l_l^{j+1}}(c_l^{j+1}) \cap (\Omega^{j+1})^c \neq \emptyset$, and $(\int_{\mathbb{R}^n} \eta_l^{j+1}(x)dx \geq (l_l^{j+1})^n)$

$$||\nabla\phi||_{L^\infty} \leq \frac{c}{l_k^j (l_l^{j+1})^n} \leq \frac{c}{(l_l^{j+1})^{n+1}}.$$

Therefore we can argue as in the proof of the Calderon-Zygmund decomposition (Lemma 4.3.5) to get for $z \in B_{C_1 l_l^{j+1}}(c_l^{j+1}) \cap (\Omega^{j+1})^c$

$$|b_{l,k}| = |\int_{\mathbb{R}^n} b_l^{j+1}(x)\phi(x)dx| \leq c(b_l^{j+1})^\star(z) \leq c2^j, \tag{4.150}$$

where we used (4.113) (note that $|z - c_l^{j+1}| \geq c l_l^{j+1}$ by our choices) in the last inequality. Altogether this shows that

$$|B_{j,k}(x)| \leq c2^j + \sum_{l=1}^{\infty} |b_{l,k}|\eta_l^{j+1}(x) \leq c2^j. \tag{4.151}$$

Therefore if we define

$$\tilde{b}_{j,k}(x) = \frac{1}{\lambda_{j,k}} B_{j,k}(x), \tag{4.152}$$

with $\lambda_{j,k}$ as in (4.137), we see that $\operatorname{spt}\tilde{b}_{j,k} \subset B_k^j$, $\int_{\mathbb{R}^n} \tilde{b}_{j,k}(x)dx = 0$ (here we use (4.146)) and

$$||\tilde{b}_{j,k}||_{L^\infty} \leq \frac{C2^j}{\lambda_{j,k}} \leq \frac{1}{|B_k^j|}. \tag{4.153}$$

This shows that the $\tilde{b}_{j,k}$'s are $\mathcal{H}^1$-atoms and from (4.149) we get

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=1}^{\infty} \lambda_{j,k}\tilde{b}_{j,k}. \tag{4.154}$$

Together with (4.141) this proves the Theorem. □

In the rest of this section we want to show how one can use the atomic decomposition to extend the L^p-theory to the situation where the right hand side is only in $\mathcal{H}^1$. For sake of simplicity we restrict ourselves to the case of the Laplace operator (for the general case see [33]).

Lemma 4.3.7. *Let* $\Gamma(x) = \frac{1}{n(2-n)\omega_n}|x|^{2-n}$*, resp.* $\Gamma(x) = -\frac{1}{2\pi}\ln(\frac{1}{|x|})$*, be the fundamental solution of* Δ *for* $n \geq 3$*, resp.* $n = 2$*. Defining* $K_{ij}(x) = \partial_i\partial_j\Gamma(x)$*, for every* $i, j \in \{1, \ldots, n\}$*, we have that* *(i)* $||K_{ij} \star f||_{L^2} \le c||f||_{L^2}$, *for every* $f \in L^2(\mathbb{R}^n)$ *and*

(ii) $\int_{2|y|\le|x|} |K_{ij}(x-y) - K_{ij}(x)|dx \le c$.

Proof. (i) is the classical L^2 estimate for the fundamental solution of Δ and can be found in [14]. For (ii) we note that $|\partial_l K_{ij}| \le \frac{c}{|x|^{n+1}}$ for every $i,j,l \in \{1,\dots,n\}$. Therefore we get for $2|y| \le |x|$ with the help of the mean value theorem that

$$|K_{ij}(x-y) - K_{ij}(x)| \le \frac{c|y|}{|x|^{n+1}}. \tag{4.155}$$

Using this we estimate

$$\int_{2|y|\le|x|} |K_{ij}(x-y) - K_{ij}(x)|dx \le c|y| \int_{2|y|}^{\infty} r^{-2} dr \le c. \tag{4.156}$$

□

Now we can prove the regularity result for equations of the form $\Delta u = f \in \mathcal{H}^1$.

Theorem 4.3.8. *Let* $f \in \mathcal{H}^1(\mathbb{R}^n)$ *and let* $u \in W^{2,1}(\mathbb{R}^n)$ *be a solution of* $\Delta u = f$, *then we have*

$$||\nabla^2 u||_{L^1(\mathbb{R}^n)} \le c||f||_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{4.157}$$

Proof. From the above considerations it is easy to see that the result follows if we show that

$$||K_{ij} \star f||_{L^1(\mathbb{R}^n)} \le c||f||_{\mathcal{H}^1(\mathbb{R}^n)}, \tag{4.158}$$

for every $i,j \in \{1,\dots,n\}$.
To see this we first consider an $\mathcal{H}^1$-atom a which is supported in a ball B_r. From (i) of Lemma 4.3.7 we get that

$$\begin{aligned} ||K_{ij} \star a||_{L^2}^2 &\le c||a||_{L^2(B_r)}^2 \\ &\le c|B_r|||a||_{L^\infty}^2 \\ &\le \frac{c}{|B_r|}, \end{aligned} \tag{4.159}$$

where we used the properties of an atom. From Hölder‘s inequality we then get

$$\begin{aligned} ||K_{ij} \star a||_{L^1(B_{2r})} &\le ||K_{ij} \star a||_{L^2} \sqrt{|B_{2r}|} \\ &\le c. \end{aligned} \tag{4.160}$$

Now we note that

$$K_{ij} \star a(x) = \int_{B_r} (K_{ij}(x-y) - K_{ij}(x))a(y)dy, \tag{4.161}$$

where we used the cancellation property of an atom. From this it follows that

$$\begin{aligned} \int_{B_{2r}^c} |K_{ij} \star a|(x)dx &\le \int_{B_{2r}^c} \int_{B_r} |K_{ij}(x-y) - K_{ij}(x)||a(y)|dydx \\ &= \int_{B_r} |a(y)|(\int_{B_{2r}^c} |K_{ij}(x-y) - K_{ij}(x)|dx)dy. \end{aligned} \tag{4.162}$$

Since $|x| \geq 2r$ and $|y| \leq r$ we can apply (ii) from Lemma 4.3.7 to get

$$\int_{B_{2r}^c} |K_{ij} \star a|(x)dx \leq c \int_{B_r} |a(y)|dy \leq c. \tag{4.163}$$

Combining (4.160) amd (4.163) this shows that

$$\int_{\mathbb{R}^n} |K_{ij} \star a|(x)dx \leq c. \tag{4.164}$$

Now we use Theorem 4.3.6 to get $f = \sum_k \lambda_k a_k$ with $\sum_k |\lambda_k| \leq c||f||_{\mathcal{H}^1}$. Then we have

$$\begin{aligned} ||K_{ij} \star f||_{L^1} &\leq c \sum_k |\lambda_k| \int_{\mathbb{R}^n} |K_{ij} \star a_k|(x)dx \\ &\leq c \sum_k |\lambda_k| \\ &\leq c||f||_{\mathcal{H}^1}. \end{aligned} \tag{4.165}$$

This proves (4.158) and therefore the Theorem. □

Remark 4.3.9. *One can actually show that* $\nabla^2 u \in \mathcal{H}^1(\mathbb{R}^n)$ *(see [33]).*

Chapter 5

Lorentz spaces

In this chapter we study the so called Lorentz spaces. These are interpolation spaces of the classical L^p-spaces and they are very useful when one needs optimal Sobolev embedding results.

5.1 Definition and basic properties

Definition 5.1.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable, $s > 0$ and let $f_\star(s) = \lambda_f(s) = |\{x \in \mathbb{R}^n | |f(x)| > s\}|$ be the distribution function of f. Then we define the non-increasing rearrangement $f^\star$ of f by*

$$f^\star(t) = \inf\{s > 0 | f_\star(s) \le t\}. \tag{5.1}$$

It is easy to see that both $f_\star$ and $f^\star$ are non-increasing and that $f_\star$ is continuous from the right.

Lemma 5.1.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then we have:*

(i) $s < f^\star(t) \Leftrightarrow t < f_\star(s)$,

(ii) $(f^\star)_\star = f_\star$,

(iii) $f^\star = (f_\star)_\star$,

(iv) $f^\star$ *is continuous from the right and*

(v) $\int_0^\infty f^\star(t)dt = \int_0^\infty f_\star(t)dt = \int_{\mathbb{R}^n} |f(x)|dx$ *and*

(vi) $f_\star(f^\star(t)) \le t$ *for all* $t > 0$.

Proof. First we prove (i). For $t > 0$ we define $M_t = \{\sigma > 0 | f_\star(\sigma) \le t\}$. Then it is easy to see that $s < f^\star(t) = \inf M_t$ implies $s \notin M_t$ and therefore $f_\star(s) > t$. For the other implication in (i) we note that M_t is an interval from somewhere to infinity. Moreover, since $f_\star$ is right continuous, M_t is left-closed if $\inf M_t \ne 0$. With the help of this we see that if $t < f_\star(s)$ then $s \notin M_t$ and therefore $s < \inf M_t = f^\star(t)$. This proves (i).
For (ii) we note that (i) implies that $\{f^\star > s\} = (0, f_\star(s))$ and therefore $(f^\star)_\star(s) = |(0, f_\star(s))| = f_\star(s)$ for every $s > 0$.
For (iii) we use again (i) to see that $\{f_\star > t\} = (0, f^\star(t))$ and then we argue as in the prove

of (ii).
(iv) follows from (iii) and the fact that for a measurable function g we have that $g_\star$ is continuous from the right.
(v) follows again from (iii) and Lemma 4.1.1.
Finally (vi) follows from the continuity from the right of $f_\star$. □

In the next Lemma we prove the Hardy-Littlewood-Polya inequality.

Lemma 5.1.3. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \int_0^\infty f^\star(t)g^\star(t)dt. \tag{5.2}$$

Proof. Using Fubini's Theorem we calculate

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)g(x)|dx &= \int_{\mathbb{R}^n} (\int_{0<r<f(x)} \int_{0<s<g(x)} drds)dx \\
&\leq \int_{(0,\infty)^2} |\{f > r\} \cap \{g > s\}|drds \\
&\leq \int_{(0,\infty)^2} \min(|\{f > r\}|, |\{g > s\}|)drds \\
&= \int_{(0,\infty)^2} \min(f_\star(r), g_\star(s))drds \\
&= \int_{(0,\infty)^2} |\{f^\star > r\} \cap \{g^\star > s\}|drds \\
&= \int_0^\infty (\int_{0<r<f^\star(t)} \int_{0<s<g^\star(t)} drds)dt \\
&= \int_0^\infty f^\star(t)g^\star(t)dt,
\end{aligned}$$

where we used that

$$|\{f^\star > r\} \cap \{g^\star > s\}| = |(0, f_\star(r)) \cap (0, g_\star(r))| = \min(f_\star(r), g_\star(s)),$$

which follows from the proof of Lemma 5.1.2. □

Next we define for $f : \mathbb{R}^n \to \mathbb{R}$ measurable

$$f^{\star\star}(t) = \frac{1}{t} \int_0^t f^\star(s)ds. \tag{5.3}$$

It is easy to see that for $f \in L^1(\mathbb{R}^n)$ we have that $f^{\star\star}$ is continuous, non-increasing and $f^{\star\star} \geq f^\star$.
Now we are able to define the Lorentz spaces.

Definition 5.1.4. *Let $1 \leq p, q \leq \infty$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then f is in the Lorentz space $L^{p,q}(\mathbb{R}^n)$ iff*

$$\begin{aligned}
||f||_{L^{p,q}(\mathbb{R}^n)} &= (\int_0^\infty (t^{\frac{1}{p}} f^{\star\star}(t))^q \frac{dt}{t})^{\frac{1}{q}}, \quad \text{if} \quad 1 \leq q < \infty, \\
||f||_{L^{p,\infty}(\mathbb{R}^n)} &= \sup\nolimits_{t>0} t^{\frac{1}{p}} f^{\star\star}(t)
\end{aligned} \tag{5.4}$$

is finite.

Next we prove the so called Hardy inequality.

Lemma 5.1.5. *Let* $q \geq 1$, $r > 0$ *and* $g : (0, \infty) \to [0, \infty)$, *then*

$$(\int_0^\infty (\int_0^t g(u)du)^q t^{-r-1} dt)^{\frac{1}{q}} \leq c(\int_0^\infty (ug(u))^q u^{-r-1} du)^{\frac{1}{q}}, \tag{5.5}$$

$$(\int_0^\infty (\int_t^\infty g(u)du)^q t^{r-1} dt)^{\frac{1}{q}} \leq c(\int_0^\infty (ug(u))^q u^{r-1} du)^{\frac{1}{q}}, \tag{5.6}$$

Proof. First of all we note the following form of Jensen's inequality: Suppose μ is a finite measure on the space M, f a μ-integrable function and φ a convex function whose domain includes the range of f, then

$$\varphi(\frac{\int_M f(t)d\mu(t)}{\mu(M)}) \leq \frac{\int_M \varphi(f(t))d\mu(t)}{\mu(M)}. \tag{5.7}$$

Now we let $\varphi(x) = |x|^q$ and $d\mu(u) = u^{\frac{r}{q}-1}du$ and insert this into (5.7) to get

$$\begin{aligned}(\int_0^t g(u)du)^q =& (\int_0^t g(u)u^{1-\frac{r}{q}} d\mu(u))^q \\ \leq& (\frac{q}{r})^{q-1} t^{r-\frac{r}{q}} \int_0^t (g(u))^q u^{q-r-1+\frac{r}{q}} du,\end{aligned} \tag{5.8}$$

where we used that $\mu((0,t)) = \frac{q}{r} t^{\frac{r}{q}}$. Thus we have

$$\begin{aligned}\int_0^\infty (\int_0^t g(u)du)^q t^{-r-1} dt &\leq c \int_0^\infty t^{-1-\frac{r}{q}} (\int_0^t g(u)^q u^{q-r-1-\frac{r}{q}} du)dt \\ &= c \int_0^\infty (ug(u))^q u^{-r-1+\frac{r}{q}} (\int_u^\infty t^{-1-\frac{r}{q}} dt)du \\ &= c \int_0^\infty \int_0^\infty (ug(u))^q u^{-r-1} du.\end{aligned} \tag{5.9}$$

Finally we show how (5.5) implies (5.6). Namely we apply (5.5) to the function $g_1(u) = \frac{1}{u^2} g(u^{-1})$. This gives

$$\begin{aligned}\int_0^\infty (\int_s^\infty g(v)dv)^q s^{r-1} ds &= \int_0^\infty (\int_{\frac{1}{t}}^\infty g(v)dv)^q t^{-r-1} dt \\ &= \int_0^\infty (\int_0^t g_1(u)du)^q t^{-r-1} dt \\ &\leq c \int_0^\infty (ug_1(u))^q u^{-r-1} du \\ &= c \int_0^\infty (vg(v))^q v^{r-1} dv.\end{aligned} \tag{5.10}$$

□

Remark 5.1.6. *With the help of the Hardy inequality (and the fact that* $f^{\star\star} \geq f^\star$*) it is easy to see that if we define the Lorentz spaces with the functions* $f^\star$ *instead of* $f^{\star\star}$ *we get equivalent* *norms. The only case which does not directly follow from Hardy's inequality is the one for $q = \infty$. In this situation we estimate by hand to get for every $t > 0$*

$$
\begin{aligned}
t^{\frac{1}{p}} f^\star(t) \le t^{\frac{1}{p}} f^{\star\star}(t) = t^{\frac{1}{p}-1} \int_0^t f^\star(s)ds \\
\le \sup_{s>0}(s^{\frac{1}{p}} f^\star(s)) t^{\frac{1}{p}-1} \int_0^t s^{-\frac{1}{p}} ds \\
= c \sup_{s>0}(s^{\frac{1}{p}} f^\star(s)).
\end{aligned} \tag{5.11}
$$

In the next Lemma we show that the Lorentz spaces are extensions of the classical L^p-spaces.

Lemma 5.1.7. *Let $1 < p < \infty$ and $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then we have*

$$c_1 ||f||_{L^p(\mathbb{R}^n)} \le ||f||_{L^{p,p}(\mathbb{R}^n)} \le c_2 ||f||_{L^p(\mathbb{R}^n)} \quad \text{and} \tag{5.12}$$

$$||f||_{L^{1,\infty}(\mathbb{R}^n)} = ||f||_{L^1(\mathbb{R}^n)}. \tag{5.13}$$

Proof. We calculate with the help of Lemma 4.1.1 and Lemma 5.1.2, (ii)

$$
\begin{aligned}
||f||^p_{L^p} =& p \int_0^\infty t^{p-1} f_\star(t) dt \\
=& \int_0^\infty (f^\star)^p(t) dt \\
=& \int_0^\infty (\frac{1}{t^p} f^\star(t))^p \frac{dt}{t}.
\end{aligned}
$$

Combining this with Remark 5.1.6 finishes the proof of (5.12). For (5.13) we calculate (using (5.3) and Lemma 5.1.2, (v))

$$
\begin{aligned}
||f||_{L^{1,\infty}} &= \sup_{t>0} t f^{\star\star}(t) \\
&= \int_0^\infty f^\star(s) ds \\
&= ||f||_{L^1}.
\end{aligned}
$$

□

The next Lemma proves a duality result for Lorentz spaces.

Lemma 5.1.8. *Let $f \in L^{p,q}(\mathbb{R}^n)$, $g \in L^{p',q'}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ $(1 < p < \infty)$. Then we have*

$$\int_{\mathbb{R}^n} |fg|(x)dx \le ||f||_{L^{p,q}(\mathbb{R}^n)} ||g||_{L^{p',q'}(\mathbb{R}^n)}. \tag{5.14}$$

Proof. We apply Lemma 5.1.3 to get

$$\int_{\mathbb{R}^n} |fg|(x)dx \le \int_0^\infty (t^{\frac{1}{p}} f^\star(t))(t^{\frac{1}{p'}} g^\star(t)) dt. \tag{5.15}$$

Hence the desired result follows from an application of Hölder's inequality and Remark 5.1.6.

□

In the following Lemma we prove inclusion results for Lorentz spaces.

Lemma 5.1.9. *Let $1 \le p, P, q, Q \le \infty$, let $\Omega \subset \mathbb{R}^n$ be bounded and let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable. Then we have*

(i) $||f||_{L^{p,Q}(\mathbb{R}^n)} \le c||f||_{L^{p,q}(\mathbb{R}^n)}$ *if* $q < Q$, $1 < p$ *and*

(ii) $|\Omega|^{-\frac{1}{p}}||f\chi_\Omega||_{L^{p,q}(\mathbb{R}^n)} \le c|\Omega|^{-\frac{1}{P}}||f\chi_\Omega||_{L^{P,Q}(\mathbb{R}^n)}$ *if* $1 < p < P$ *and* q, Q *are abitrary.*

Proof. First we prove (i) for $Q = \infty$.

$$(t^{\frac{1}{p}} f^\star(t))^q \le c(f^\star(t))^q \int_0^t s^{\frac{q}{p}-1} ds \le c \int_0^t (s^{\frac{1}{p}} f^\star(s))^q \frac{ds}{s}. \tag{5.16}$$

This proves the desired result for $Q = \infty$ (using Remarks 5.1.6). For $Q < \infty$ we estimate

$$\int_0^\infty (t^{\frac{1}{p}} f^\star(t))^Q \frac{dt}{t} \le \sup_{s>0}(s^{\frac{1}{p}} f^\star(s))^{Q-q}||f||^q_{L^{p,q}} \le c||f||^Q_{L^{p,q}}, \tag{5.17}$$

where we used Remarks 5.1.6 and (5.16) in the last step. These two things prove (i). For (ii) we note first that $(f\chi_\Omega)_\star(s) = |\{x \in \Omega | |f|(x) > s\}| \le |\Omega|$ and hence $(f\chi_\Omega)^\star(t) = \inf\{s > 0 | (f\chi_\Omega)_\star(s) \le t\} = 0$ for $t > |\Omega|$. Hence we get from Remark 5.1.6

$$||f\chi_\Omega||_{L^{p,1}} \le c \int_0^{|\Omega|} t^{\frac{1}{p}-1}(f\chi_\Omega)^\star(t) dt \le \sup_{s>0}(s^{\frac{1}{P}}(f\chi_\Omega)^\star(s)) \int_0^{|\Omega|} t^{\frac{1}{p}-\frac{1}{P}-1} dt \le \frac{|\Omega|^{\frac{1}{p}-\frac{1}{P}}}{\frac{1}{p}-\frac{1}{P}} ||f\chi_\Omega||_{L^{P,\infty}}. \tag{5.18}$$

Since by (i) $||f\chi_\Omega||_{L^{P,\infty}} \le c||f\chi_\Omega||_{L^{P,Q}}$ for every $1 \le Q \le \infty$ and $||f\chi_\Omega||_{L^{p,1}} \ge c||f\chi_\Omega||_{L^{p,q}}$ for every $1 \le q \le \infty$ this proves the desired result. □

Nex we prove:

Lemma 5.1.10. *Let $\alpha < n$. Then we have*

$$I_\alpha(x) = |x|^{\alpha-n} \in L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n). \tag{5.19}$$

Proof. We calculate

$$(I_\alpha)_\star(s) = |\{|x|^{\alpha-n} > s\}| = |\{|x| < s^{\frac{1}{\alpha-n}}\}| = \omega_n s^{\frac{n}{\alpha-n}}$$

therefore

$$\begin{aligned}(I_\alpha)^\star(t) &= \inf\{s>0|\omega_n s^{\frac{n}{\alpha-n}} \le t\}\\ &= (\frac{t}{\omega_n})^{\frac{\alpha-n}{n}}\end{aligned}$$

and

$$(I_\alpha)^{\star\star}(t) = \frac{n\omega_n^{\frac{n-\alpha}{n}}}{\alpha} t^{\frac{\alpha-n}{n}}.$$

Hence

$$||I_\alpha||_{L^{\frac{n}{n-\alpha},\infty}} = \sup\nolimits_{t>0}(t^{\frac{n-\alpha}{n}} \frac{n\omega_n^{\frac{n-\alpha}{n}}}{\alpha} t^{\frac{\alpha-n}{n}}) = \frac{n\omega_n^{\frac{n-\alpha}{n}}}{\alpha} < \infty.$$

□

The next Lemma is a technical result which we need later on.

Lemma 5.1.11. *Let $1 \le p \le \infty$, $1 \le q < \infty$ and $f \in L^{p,q}(\mathbb{R}^n)$. Then we have*

$$||f||^q_{L^{p,q}(\mathbb{R}^n)} \ge c \int_0^\infty s^{q-1}(f_\star(s))^{\frac{q}{p}} ds. \tag{5.20}$$

Proof. We define the sets

$$\begin{aligned}\Gamma_p(f) &= \{(r,s) \in (0,\infty)^2 | f^\star(r^p) > s\}\\ &= \{(r,s) \in (0,\infty)^2 | f_\star(s) > r^p\},\end{aligned} \tag{5.21}$$

where the second equality follows from Lemma 5.1.2, (i). Using Fubini's theorem and the transformation formula we calculate

$$\begin{aligned}\int_0^\infty t^{\frac{q}{p}-1}(f^\star(t))^q dt &= p\int_0^\infty r^{q-p}(f^\star(r^p))^q r^{p-1} dr\\ &= pq\int_0^\infty (\int_0^{f^\star(r^p)} (rs)^{q-1} ds) dr\\ &= pq\int_{\Gamma_p(f)} (rs)^{q-1} ds dr\\ &= pq\int_0^\infty (\int_0^{(f_\star(s))^{\frac{1}{p}}} (rs)^{q-1} dr) ds\\ &= p\int_0^\infty s^{q-1}(f_\star(s))^{\frac{q}{p}} ds.\end{aligned} \tag{5.22}$$

Combining this with Remark 5.1.6 the result follows. □

Lemma 5.1.12. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then we have for every $s > 0$*

$$(Mf)_\star(Cs) \le \frac{1}{s}\int_{\{f>s\}} f(x)dx, \tag{5.23}$$

where C depends only on n.

Proof. Without loss of generality we assume $f \geq 0$. We set $C = 5^n + 1$. For every $z \in \{Mf > Cs\}$ there exists a ball $B_z = B_{r_z}(z)$ such that

$$Cs|B_z| \leq \int_{B_z} f(x)dx \leq \int_{B_z \cap \{f>s\}} f(x)dx + s|B_z|,$$

so that

$$(C-1)s|B_z| \leq \int_{B_z \cap \{f>s\}} f(x)dx.$$

Using the covering Lemma of Besicovitch (see Lemma 4.1.3) we obtain a sequence of pairwise disjoint balls $B_j = B_{r_j}(z_j) \in \{B_z | z \in \{Mf > Cs\}\}$ such that $\{Mf > Cs\} \subset \cup_j B_{5r_j}(z_j)$. Moreover we have

$$\begin{aligned} s(Mf)_\star(Cs) &= s|\{Mf > Cs\}| \\ &\leq s5^n \sum_l |B_j| \\ &\leq \frac{5^n}{C-1} \sum_j \int_{B_j \cap \{f>s\}} f(x)dx \\ &\leq \int_{\{f>s\}} f(x)dx. \end{aligned}$$

□

Lemma 5.1.13. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then we have for every $t > 0$*

$$(Mf)^\star(t) \leq cf^{\star\star}(t), \tag{5.24}$$

where c only depends on n.

Proof. Using Lemma 5.1.12 we have

$$\begin{aligned} f^{\star\star}(t)(Mf)_\star(Cf^{\star\star}(t)) &\leq \int_{\{f>f^{\star\star}(t)\}} f(x)dx \\ &\leq \int_{\{f>f^\star(t)\}} f(x)dx \\ &\leq \int_0^t f^\star(s)ds \\ &= tf^{\star\star}(t), \end{aligned}$$

where we used that

$$\int_{\{f>f^{\star\star}(t)\}} f(x)dx = \int_0^\infty (f\chi_{\{f>f^\star(t)\}})^\star(t)dt$$

and $(f\chi_{\{f>f^\star(t)\}})_\star(s) \leq f_\star(\max(s, f^\star(t)))$ and therefore $(f\chi_{\{f>f^\star(t)\}})^\star(\tau) \leq \chi_{\{\tau<t\}} f^\star(\tau)$. Altogether this shows that

$$Cf^{\star\star}(t) \in \{s \geq 0 | (Mf)_\star(s) \leq t\}$$

and therefore

$$(Mf)^\star(t) \leq Cf^{\star\star}(t).$$

□

Corollary 5.1.14. *Let* $1 < p < \infty$ *and* $f \in L^{p,q}(\mathbb{R}^n)$*. Then we have*

$$||Mf||_{L^{p,q}(\mathbb{R}^n)} \leq c||f||_{L^{p,q}(\mathbb{R}^n)}. \tag{5.25}$$

Proof. This follows directly from Lemma 5.1.13 and Remark 5.1.6. This can also be proved by combining the Hardy-Littlewood Theorem 4.1.4 and the Marcinkiewicz interpolation Theorem 5.2.1. □

Lemma 5.1.15. *Let* $1 < p < \infty$, $1 \leq q < \infty$, $f \in L^{p,q}(\mathbb{R}^n)$ *and* ψ *a standard mollifier. Then* $\Psi \star f \to f$ *in* $L^{p,q}(\mathbb{R}^n)$.

Proof. By Luzin's Theorem we can decompose f into a part f_1 which is continuous with compact support and a part f_2 which has a small $L^{p,q}$-norm. For the part f_1 we apply the classical convergence results in L^p-spaces and the embedding lemma 5.1.9 to get the desired convergence property. For f_2 we use the estimate

$$\psi \star f_2(x) \leq cMf_2(x),$$

which, together with Lemma 5.1.14, implies

$$||\psi \star f_2||_{L^{p,q}} \leq c||Mf_2||_{L^{p,q}} \leq c||f_2||_{L^{p,q}}$$

and therefore the Lemma. □

Lemma 5.1.16. *Let* $E \subset \mathbb{R}^n$ *be Lebesgue measurable. Then we have*

$$\int_E |x|^{1-n} \leq c|E|^{\frac{1}{n}}. \tag{5.26}$$

Proof. First of all we note that

$$\int_{B_R(0)} |x|^{1-n} = \omega_n R. \tag{5.27}$$

Next we choose a ball $B = B_R(0)$ such that $|B| = |E|$. This implies $|B\backslash E| = |B| - |B \cap E| = |E| - |E \cap B| = |E\backslash B|$. Hence we have

$$\int_{E\backslash B} |x|^{1-n} \leq \int_{E\backslash B} R^{1-n} = \int_{B\backslash E} R^{1-n} \leq \int_{B\backslash E} |x|^{1-n}.$$

Adding $\int_{E\cap B} |x|^{1-n}$ to both sides of this estimate we get

$$\int_E |x|^{1-n} \leq \int_B |x|^{1-n} \leq \omega_n R = c|B|^{\frac{1}{n}} = c|E|^{\frac{1}{n}}.$$

□

Now we are in a position to prove the first main Theorem using Lorentz spaces. Namely we show that functions whose gradient is in $L^{n,1}$ are continuous.

Theorem 5.1.17. *Let $\nabla u \in L^{n,1}(\mathbb{R}^n)$ and $B = B_r(y)$ be a ball with $z \in B$. Then we have that u is continuous and*

$$|u(z) - \frac{1}{|B|}\int_B u| \leq c||\chi_B \nabla u||_{L^{n,1}(\mathbb{R}^n)}. \tag{5.28}$$

Proof. We assume without loss of generality that $\operatorname{spt} u$ is compact, $z = 0$ and $B = B_1(0)$. Moreover we let Γ be the fundamental solution of the Laplacian on B which vanishes on ∂B. Hence we have $|\nabla \Gamma(x)| \leq c|x|^{1-n}$.
In a first step of the proof we now assume that $u \in C^\infty$. Then we can estimate

$$\begin{aligned}
|u(0) - \frac{1}{|B|}\int_B u| &\leq c|\int_B \Gamma(x)\Delta u(x)dx| \\
&\leq c\int_{\mathbb{R}^n} |x|^{1-n}\chi_B|\nabla u|dx \\
&= c\int_{\mathbb{R}^n} |x|^{1-n}(\int_0^{\chi_B|\nabla u|} ds)dx \\
&= c\int_0^\infty (\int_{\{\chi_B|\nabla u|>s\}} |x|^{1-n}dx)ds \\
&\leq c\int_0^\infty (\chi_B|\nabla u|)_\star^{\frac{1}{n}}(s)ds \\
&\leq c||\chi_B|\nabla u|||_{L^{2,1}},
\end{aligned} \tag{5.29}$$

where we used Lemma 5.1.16 in the second last step and Lemma 5.1.11 in the last step. This proves (5.28) for smooth functions. For general u we consider the mollifications $u_\delta = u \star \Psi_\delta$ and note that by Lemma 5.1.15 we have

$$||\nabla(u_{\delta_1} - u_{\delta_2})||_{L^{2,1}} \to 0, \tag{5.30}$$

as $\delta_1, \delta_2 \to 0$. Hence the sequence $(u_\delta)_\delta$ is a Cauchy sequence in L^∞ and therefore the limit u is continuous. □

Next we derive an improved Sobolev embedding result which is due to Poornima [26].

Theorem 5.1.18. *For $n \geq 2$ we have that $W^{1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1},1}(\mathbb{R}^n)$ continuously. That is for $f \in W^{1,1}(\mathbb{R}^n)$ we have*

$$\int_0^\infty t^{-\frac{1}{n}} f^\star(t)dt \leq c||f||_{W^{1,1}(\mathbb{R}^n)}. \tag{5.31}$$

Proof. We let $\varphi : \mathbb{R} \to [0,1]$ be a C^1 function with $\varphi(x) = 1$ for $x > 1$ and $\varphi(x) = 0$ for $x < 0$. Then we define for $\varepsilon > 0$, $\lambda \geq 0$

$$\varphi_\varepsilon^\lambda(x) = \varphi(\frac{x-\lambda}{\varepsilon}).$$

We have

$$\varphi_\varepsilon^\lambda(x) = 0 \quad \text{for} \quad x < \lambda$$
$$\varphi_\varepsilon^\lambda(x) = 1 \quad \text{for} \quad x > \lambda + \varepsilon.$$

Moreover we have

$$||\nabla \varphi_\varepsilon^\lambda||_{L^\infty} \le \frac{c}{\varepsilon}.$$

Now we claim that for every $f \in C_c^1(\mathbb{R}^n)$ we have

$$\int_0^\infty ||\varphi_\varepsilon^\lambda \circ f||_{W^{1,1}} d\lambda \le c||f||_{W^{1,1}}. \tag{5.32}$$

To see this we estimate

$$\int_0^\infty ||\varphi_\varepsilon^\lambda \circ f||_{L^1} d\lambda \le \int_0^\infty ||\chi_{\{|f|>\lambda\}}||_{L^1} d\lambda = \int_0^\infty f_\star(\lambda) d\lambda = ||f||_{L^1}.$$

Moreover we have

$$|\nabla(\varphi_\varepsilon^\lambda \circ f)| = |(\varphi_\varepsilon^\lambda)' \circ f \nabla f| \le \frac{c}{\varepsilon}\chi_{\{\lambda < f < \lambda+\varepsilon\}}|\nabla f|$$

and hence

$$\int_0^\infty ||\nabla(\varphi_\varepsilon^\lambda \circ f)||_{L^1} d\lambda \le \frac{c}{\varepsilon}\int_{\mathbb{R}^n} |\nabla f|(\int_{f-\varepsilon}^f d\lambda) dx \le c||\nabla f||_{L^1}.$$

This proves (5.32). Next we claim that for every $f \in C_c^1(\mathbb{R}^n)$ we have

$$\int_0^\infty (f_\star(s))^{\frac{n-1}{n}} ds \le c||f||_{W^{1,1}(\mathbb{R}^n)}. \tag{5.33}$$

To show this estimate we define $f_\star^+(s) = |\{f > s\}|$ and $f_\star^- = |\{-f > s\}|$. Since $f_\star \le f_\star^+ + f_\star^-$ we have

$$(f_\star)^{\frac{n-1}{n}}(s) \le (f_\star^+)^{\frac{n-1}{n}}(s) + (f_\star^-)^{\frac{n-1}{n}}(s)$$

and hence it suffices to prove (5.33) for $f_\star^+$ and $f_\star^-$. From the classical Sobolev embedding we know that

$$||\varphi_\varepsilon^\lambda \circ f||_{L^{\frac{n}{n-1}}} \le c||\varphi_\varepsilon^\lambda \circ f||_{W^{1,1}}$$

and therefore by (5.32) we have

$$\int_0^\infty (\int_{\mathbb{R}^n} |\varphi_\varepsilon^\lambda \circ f|^{\frac{n}{n-1}}(x) dx)^{\frac{n-1}{n}} d\lambda \le c||f||_{W^{1,1}}.$$

If we define $\psi_\varepsilon^\lambda = \int_{\mathbb{R}^n} |\varphi_\varepsilon^\lambda \circ f|^{\frac{n}{n-1}}(x) dx$ we have

$$\int_0^\infty (\psi_\varepsilon^\lambda)^{\frac{n-1}{n}} d\lambda \le c||f||_{W^{1,1}}.$$

For any $\lambda > 0$ we see that

$$\varphi_\varepsilon^\lambda \circ f \to \chi_{\{f>\lambda\}}$$

as $\varepsilon \to 0$. If we let $\operatorname{spt} f = K$ we also have that

$$|\varphi_\varepsilon^\lambda \circ f|^{\frac{n}{n-1}} \leq \chi_K \in L^1(\mathbb{R}^n).$$

Hence we can apply the dominated convergence theorem to get

$$\psi_\varepsilon^\lambda \to \int_{\mathbb{R}^n} \chi_{\{f>\lambda\}} = f_\star^+(\lambda)$$

and therefore

$$(\psi_\varepsilon^\lambda)^{\frac{n-1}{n}} \to (f_\star^+(\lambda))^{\frac{n-1}{n}}.$$

Hence we get from Fatout́s Lemma

$$\int_0^\infty (f_\star^+(\lambda))^{\frac{n-1}{n}} \leq c||f||_{W^{1,1}}$$

and this shows (5.33) for $f_\star^+$. The argument for $f_\star^-$ is similar and therefore we finish the proof of (5.33).
Combining (5.20) with (5.33) we get the desired result for $f \in C_c^1(\mathbb{R}^n)$. Since these functions are dense in $W^{1,1}(\mathbb{R}^n)$ and $L^{\frac{n}{n-1},1}(\mathbb{R}^n)$ (by Lemma 5.1.15) we finish the proof of the Theorem. $\square$

Remark 5.1.19. *By induction we can actually prove that for every $1 < k < n$ we have $W^{k-1,1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-k+1},1}(\mathbb{R}^n)$ with the corresponding estimate for the norms.*

Combining Theorem 5.1.17 and Theorem 5.1.18 (resp. Remark 5.1.19) we get

Corollary 5.1.20. *Every function $u \in W^{n,1}(\mathbb{R}^n)$ is continuous.*

5.2 Interpolation and PDE estimates

The next Theorem is the so called Marcinkiewicz interpolation theorem for Lorentz spaces and it is taken from [34].

Theorem 5.2.1. *Let $1 \leq r_0 < r_1 \leq \infty$, $1 \leq p_0 \neq p_1 \leq \infty$ and let T be a subadditive operator satisfying*

$$||Tf||_{L^{p_0,\infty}(\mathbb{R}^n)} \leq c||f||_{L^{r_0,1}(\mathbb{R}^n)} \quad \forall \;\; f \in L^{r_0,1}(\mathbb{R}^n), \tag{5.34}$$

$$||Tf||_{L^{p_1,\infty}(\mathbb{R}^n)} \leq c||f||_{L^{r_1,1}(\mathbb{R}^n)} \quad \forall \;\; f \in L^{r_1,1}(\mathbb{R}^n). \tag{5.35}$$

Then we have for every $1 \leq q \leq \infty$, $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ $(0 < \theta < 1)$ and every $f \in L^{r,q}(\mathbb{R}^n)$

$$||Tf||_{L^{p,q}(\mathbb{R}^n)} \leq c||f||_{L^{r,q}(\mathbb{R}^n)}, \tag{5.36}$$

where $c = c(p_i, r_i, \theta, r, p)$.

Proof. For $f \in L^{r,q}(\mathbb{R}^n)$ we define

$$f^t(x) = \chi_{\{|f|>f^\star(t^\gamma)\}} f(x) \tag{5.37}$$

and

$$f_t(x) = f(x) - f^t(x), \tag{5.38}$$

where $\gamma = \frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{r_0}-\frac{1}{r}} = \frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{r}-\frac{1}{r_1}}$. Now we have that

$$(f^t)_\star(s) = \chi_{\{s<f^\star(t^\gamma)\}} f_\star(f^\star(t^\gamma)) + (1-\chi_{\{s<f^\star(t^\gamma)\}}) f_\star(s). \tag{5.39}$$

Since $(f^t)_\star(s) \leq \min(f_\star(s), f_\star(f^\star(t^\gamma)))$ for every $s > 0$ we see that $(f^t)^\star(l) = 0$ if $l \geq t^\gamma$ (because of Lemma 5.1.2, (vi)). For $l < t^\gamma$ we easily calculate that

$$(f^t)^\star(l) \leq \chi_{\{l<t^\gamma\}} f^\star(l). \tag{5.40}$$

In the same way we see that

$$(f_t)^\star(l) \leq \chi_{\{t^\gamma \leq l\}} f^\star(l) + \chi_{\{t^\gamma > l\}} f^\star(t^\gamma). \tag{5.41}$$

Moreover we calculate

$$\begin{aligned}
||f^t||_{L^{r_0,1}} \leq & c\int_0^{t^\gamma} s^{\frac{1}{r_0}-1} f^\star(s)ds \\
\leq & c(t)||f||_{L^{r,\infty}} \\
\leq & c(t)||f||_{L^{r,q}}
\end{aligned}$$

and

$$||f_t||_{L^{r_1,1}} \leq cf^\star(t^\gamma)\int_0^{t^\gamma} s^{\frac{1}{r_1}-1} + c||f||_{L^{r,\infty}} \int_{t^\gamma}^\infty s^{\frac{1}{r_1}-\frac{1}{r}-1} ds \leq c(t) + c(t)||f||_{L^{r,q}}.$$

Since T is subadditive we have for almost every y that

$$|Tf(y)| = T(f^t + f_t)(y)| \leq |Tf^t(y)| + |Tf_t(y)|.$$

Thus we get for $l > 0$

$$\{|Tf(y)| > (Tf^t)^\star(l) + (Tf_t)^\star(l)\} \subset \{|Tf^t(y)| > (Tf^t)^\star(l)\} \cup \{|Tf_t(y)| > (Tf_t)^\star(l)\}$$

and hence

$$\begin{aligned}
|\{|Tf(y)| > (Tf^t)^\star(l) + (Tf_t)^\star(l)\}| \leq & |\{|Tf^t(y)| > (Tf^t)^\star(l)\}| + |\{|Tf_t(y)| > (Tf_t)^\star(l)\}| \\
= & (Tf^t)_\star((Tf^t)^\star(l)) + (Tf_t)_\star((Tf_t)^\star(l)) \\
\leq & 2l
\end{aligned} \tag{5.42}$$

Therefore we have

$$(Tf)^\star(2l) \leq (Tf^t)^\star(l) + (Tf_t)^\star(l). \tag{5.43}$$

Now we consider three cases:

1) $r_1 < \infty$ and $q < \infty$
We apply (5.43) with $l = t$ to get

$$
\begin{aligned}
||Tf||^q_{L^{p,q}} &= \int_0^\infty t^{\frac{q}{p}-1}((Tf)^\star)^q(t)dt \\
&= c\int_0^\infty t^{\frac{q}{p}-1}((Tf)^\star)^q(2t)dt \\
&\le c\int_0^\infty t^{\frac{q}{p}-1}((Tf^t)^\star)^q(t)dt + c\int_0^\infty t^{\frac{q}{p}-1}((Tf_t)^\star)^q(t)dt. \qquad (5.44)
\end{aligned}
$$

From (5.34) and (5.35) we get

$$
\begin{aligned}
t^{\frac{1}{p_0}}(Tf^t)(t) \le& c||f^t||_{L^{r_0,1}}, \qquad (5.45)\\
t^{\frac{1}{p_1}}(Tf_t)(t) \le& c||f_t||_{L^{r_1,1}}. \qquad (5.46)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^\infty t^{\frac{q}{p}-1}((Tf^t)^\star)^q(t)dt \le& c\int_0^\infty (t^{\frac{1}{p}-\frac{1}{p_0}}||f^t||_{L^{r_0,1}})^q\frac{dt}{t}\\
\le& c\int_0^\infty (t^{\frac{1}{p}-\frac{1}{p_0}}(\int_0^{t^\gamma} s^{\frac{1}{r_0}-1}f^\star(s)ds))^q\frac{dt}{t}\\
=& c\int_0^\infty (u^{\frac{1}{r}-\frac{1}{r_0}}(\int_0^{u} s^{\frac{1}{r_0}-1}f^\star(s)ds))^q\frac{du}{u}\\
\le& c\int_0^\infty (u^{\frac{1}{r_0}}f^\star(u))^q u^{\frac{q}{r}-\frac{q}{r_0}-1}du\\
\le& c||f||^q_{L^{r,q}}, \qquad (5.47)
\end{aligned}
$$

where we used (5.5) in the second last step. Moreover we estimate

$$
\begin{aligned}
\int_0^\infty t^{\frac{q}{p}-1}((Tf_t)^\star)^q(t)dt \le& c\int_0^\infty (t^{\frac{1}{p}-\frac{1}{p_1}}||f_t||_{L^{r_1,1}})^q\frac{dt}{t}\\
\le& c\int_0^\infty (t^{\frac{1}{p}-\frac{1}{p_1}}(\int_0^{t^\gamma} s^{\frac{1}{r_1}-1}f^\star(t^\gamma)ds))^q\frac{dt}{t}\\
&+ c\int_0^\infty (t^{\frac{1}{p}-\frac{1}{p_1}}(\int_{t^\gamma}^{\infty} s^{\frac{1}{r_1}-1}f^\star(s)ds))^q\frac{dt}{t}\\
=& c\int_0^\infty (t^{\frac{1}{p}-\frac{1}{p_1}}f^\star(t^\gamma)t^{\frac{\gamma}{r_1}})^q\frac{dt}{t}\\
&+ c\int_0^\infty (t^{\frac{1}{p}-\frac{1}{p_1}}(\int_{t^\gamma}^{\infty} s^{\frac{1}{r_1}-1}f^\star(s)ds))^q\frac{dt}{t}\\
=& c\int_0^\infty (u^{\frac{1}{r}}f^\star(u))^q\frac{du}{u} + c\int_0^\infty (u^{\frac{1}{r}-\frac{1}{r_1}}(\int_u^\infty s^{\frac{1}{r_1}-1}f^\star(s)ds))^q\frac{du}{u}\\
\le& c||f||_{L^{r,q}} + c\int_0^\infty (u^{\frac{1}{r_1}}f^\star(u))^q u^{\frac{q}{r}-\frac{q}{r_1}-1}du\\
\le& c||f||_{L^{r,q}}, \qquad (5.48)
\end{aligned}
$$

where we used (5.6) in the second last step. Altogether this finishes the proof in the first case.

2) $r_1 < \infty$ and $q = \infty$
In the case we argue as above and get (using (5.43))

$$\begin{aligned} t^{\frac{1}{p}}(Tf)^\star(t) \leq& ct^{\frac{1}{p}-\frac{1}{p_0}}\int_0^{t^\gamma} s^{\frac{1}{r_0}-1}f^\star(s)ds + ct^{\frac{1}{p}-\frac{1}{p_1}}\int_0^{t^\gamma} s^{\frac{1}{r_1}-1}f^\star(t^\gamma)ds \\ &+ ct^{\frac{1}{p}-\frac{1}{p_1}}\int_{t^\gamma}^\infty s^{\frac{1}{r_1}-1}f^\star(s)ds \\ \leq& c||f||_{L^{r,\infty}}t^{\frac{1}{p}-\frac{1}{p_0}}\int_0^{t^\gamma} s^{\frac{1}{r_0}-\frac{1}{r}-1}ds + cf^\star(t^\gamma)t^{\frac{1}{p}-\frac{1}{p_1}}t^{\frac{\gamma}{r_1}} \\ &+ c||f||_{L^{r,\infty}}t^{\frac{1}{p}-\frac{1}{p_1}}\int_{t^\gamma}^\infty s^{\frac{1}{r_1}-\frac{1}{r}-1}ds \\ \leq& c||f||_{L^{r,\infty}}. \end{aligned} \tag{5.49}$$

3) $r_1 = \infty$ and $q = \infty$
In this case we argue as above but this time we have to use that $||f_t||_{L^{\infty,\infty}} \leq f^\star(t^\gamma)$. (Exercise!)

□

In the following we prove some PDE-estimates involving Lorentz-spaces. Before doing this we need the following Lemma (see [19]).

Lemma 5.2.2. *Let $\Omega \subset \mathbb{R}^n$ be bounded with C^1-boundary, let $g = (g_1, g_2) \in L^1(\Omega, \mathbb{R}^2)$ and let α be a solution of*

$$\begin{aligned} \Delta\alpha &= \operatorname{div} g \quad \text{in} \;\; \Omega, \\ \alpha &= 0 \quad \text{on} \;\; \partial\Omega. \end{aligned} \tag{5.50}$$

Then we have that the operator

$$P(g) = \nabla\alpha \tag{5.51}$$

is continuous between $L^{p,q}(\Omega, \mathbb{R}^2)$, $1 < p < \infty$, $1 \leq q \leq \infty$, and itself.

Proof. From standard L^p-theory we know that P is continuous between $L^p(\Omega, \mathbb{R}^2)$, $1 < p < \infty$, and itself. Therefore we can apply Theorem 5.2.1 to get the desired result. □

Lemma 5.2.3. *Let $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,q_2}(\mathbb{R}^n)$ with $\frac{1}{p_1} + \frac{1}{p_2} > 1$. Then $h = f \star g \in L^{r,s}(\mathbb{R}^n)$ where $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and s is a number such that $\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}$. Moreover we have*

$$||h||_{L^{r,s}(\mathbb{R}^n)} \leq c||f||_{L^{p_1,q_1}(\mathbb{R}^n)}||g||_{L^{p_2,q_2}(\mathbb{R}^n)}. \tag{5.52}$$

Proof. See [41]. □

The next three theorems are taken from [19] (see also [1] and [12]).

Theorem 5.2.4. *Let $\Omega \subset \mathbb{R}^2$ be open with $\partial\Omega \in C^1$. Let $f \in L^1(\Omega)$ and let φ be a solution of*

$$\begin{aligned} -\Delta\varphi &= f \quad \text{in} \;\; \Omega, \\ \varphi &= 0 \quad \text{on} \;\; \partial\Omega, \end{aligned} \tag{5.53}$$

then $\nabla\varphi \in L^{2,\infty}(\Omega, \mathbb{R}^2)$ and

$$||\nabla\varphi||_{L^{2,\infty}(\Omega)} \leq c(\Omega)||f||_{L^1(\Omega)}. \tag{5.54}$$

Proof. We consider $\overline{f} \in L^1(\mathbb{R}^2)$ which we obtain by extending f by 0 outside of Ω and we define

$$\psi(x) = \int_{\mathbb{R}^2} K(x-y)\overline{f}(y)dy,$$

where $K(x) = \frac{1}{2\pi}\ln(\frac{1}{|x|})$. Then we know that

$$-\Delta\psi = \overline{f}$$

in $\mathbb{R}^2$ and

$$\nabla\psi = \int_{\mathbb{R}^2} \nabla K(x-y)\overline{f}(y)dy.$$

Since $\overline{f} \in L^1(\mathbb{R}^2) = L^{1,\infty}(\mathbb{R}^2)$ and $|\nabla K|(x-y) \leq \frac{c}{|x-y|} \in L^{2,\infty}$ (see Lemma 5.1.10) we can apply Lemma 5.2.3 to get that $\nabla\psi \in L^{2,\infty}$ and

$$||\nabla\psi||_{L^{2,\infty}} \leq c||\overline{f}||_{L^1} = c||f||_{L^1(\Omega)}.$$

Since $\nabla\varphi = P(\nabla\psi|_\Omega)$ we can apply Lemma 5.2.2 to conclude the proof of the Theorem. □

In the following Theorem we improve Wente‘s inequality.

Theorem 5.2.5. *Let $\Omega \subset \mathbb{R}^2$ be open with C^1-boundary, let $f \in \mathcal{H}^1(\mathbb{R}^2)$ and let φ be a solution of*

$$\begin{aligned} -\Delta\varphi &= f \quad in \ \ \Omega, \\ \varphi &= 0 \quad on \ \ \partial\Omega, \end{aligned} \tag{5.55}$$

then $\nabla\varphi \in L^{2,1}(\Omega, \mathbb{R}^2)$ and

$$||\nabla\varphi||_{L^{2,1}(\Omega)} \leq c(\Omega)||f||_{\mathcal{H}^1(\mathbb{R}^2)}. \tag{5.56}$$

Proof. We let

$$\psi(x) = \int_{\mathbb{R}^2} K(x-y)f(y)dy,$$

where $K(x) = \frac{1}{2\pi}\ln(\frac{1}{|x|})$. Then we know that

$$-\Delta\psi = f$$

on $\mathbb{R}^2$. By Theorem 4.3.8 we know that $\psi \in W^{2,1}(\mathbb{R}^2)$ with

$$||\nabla^2\psi||_{L^1} \leq c||f||_{\mathcal{H}^1}.$$

Hence we get from Theorem 5.1.18 that $\nabla\psi \in L^{2,1}(\mathbb{R}^2)$ with

$$||\nabla\psi||_{L^{2,1}} \leq c||f||_{\mathcal{H}^1}.$$

Using again that $\nabla\varphi = P(\nabla\psi|_\Omega)$ we can apply Lemma 5.2.2 to conclude the proof of the Theorem. □

The next theorem is also an improvement of Wente's inequality. This theorem has recently been used T. Rivière [29] in his study of Willmore surfaces.

Theorem 5.2.6. *Let $\Omega \subset \mathbb{R}^2$ be open with C^2-boundary, let a, b be functions such that $\nabla a \in L^{2,\infty}(\Omega)$, $b \in W^{1,2}(\Omega)$ and let φ be a solution of*

$$\begin{aligned} -\Delta\varphi &= \{a, b\} \quad \text{in} \;\; \Omega, \\ \varphi &= 0 \quad \text{on} \;\; \partial\Omega, \end{aligned} \tag{5.57}$$

then $\varphi \in W^{1,2}(\Omega)$ and

$$||\nabla\varphi||_{L^2(\Omega)} \leq c||\nabla a||_{L^{2,\infty}(\Omega)}||\nabla b||_{L^2(\Omega)}. \tag{5.58}$$

Proof. We first prove (5.58) for $a, b \in W^{1,2}(\Omega)$. Let $U \supset \Omega$ be smooth and bounded. Moreover we let $\tilde{a}, \tilde{b} \in W_0^{1,2}(U)$ be the extensions of a, b and we let ψ be the solution of

$$\begin{aligned} -\Delta\psi &= \{\tilde{a}, \tilde{b}\} \quad \text{in} \;\; U, \\ \psi &= 0 \quad \text{on} \;\; \partial U. \end{aligned}$$

Then we he have

$$\begin{aligned} ||\nabla\psi||_{L^2}^2 &= -\int_U \psi\Delta\psi \\ &= \int_U \psi\{\tilde{a}, \tilde{b}\} \\ &= \int_U \tilde{a}\{\psi, \tilde{b}\} \\ &= -\int_U \tilde{a}\Delta\Psi \\ &= \int_U \nabla\tilde{a}\nabla\Psi \\ &\leq c||\nabla\tilde{a}||_{L^{2,\infty}}||\nabla\Psi||_{L^{2,1}} \\ &\leq c||\nabla\tilde{a}||_{L^{2,\infty}}||\nabla\tilde{b}||_{L^2}||\nabla\psi||_{L^2}, \end{aligned}$$

where we used Lemma 5.1.8, Theorem 5.2.5 and where Ψ is a solution of

$$\begin{aligned} -\Delta\Psi &= \{\psi, \tilde{b}\} \quad \text{in} \;\; U, \\ \Psi &= 0 \quad \text{on} \;\; \partial U. \end{aligned}$$

Since the extension operator is continuous from $W^{1,p}$ to $W^{1,p}$ for every $1 < p < \infty$ we can apply Theorem 5.2.1 to get

$$||\nabla\psi||_{L^2(U)} \leq c||\nabla a||_{L^{2,\infty}(\Omega)}||\nabla b||_{L^2(\Omega)}.$$

Moreover, since

$$\begin{aligned} -\Delta\varphi &= -\Delta\psi \quad \text{in} \;\; \Omega, \\ \varphi &= 0 \quad \text{on} \;\; \partial\Omega, \end{aligned}$$

we get that

$$\begin{aligned}||\nabla\varphi||_{L^2(\Omega)} &\leq ||\nabla\psi||_{L^2(U)} \\ &\leq c||\nabla a||_{L^{2,\infty}(\Omega)}||\nabla b||_{L^2(\Omega)}.\end{aligned}$$

This proves (5.58) for $a, b \in W^{1,2}(\Omega)$.
If we now only have a such that $\nabla a \in L^{2,\infty}(\Omega)$ we choose a sequence $a_k \in \cap_{1\leq p<2} W^{1,p}(\Omega)$ such that $a_k \to a$ in $W^{1,p}(\Omega)$ for every $p < 2$ and $||\nabla a_k||_{L^{2,\infty}} \leq c||a||_{L^{2,\infty}}$ (note that you can't find a sequence $a_k \in W^{1,2}$ with the above properties). Indeed you can just consider the convolution of a with a sequence of mollifiers which are in L^1 and then you can apply Lemma 5.2.3 to get the desired properties. Then we have by (5.58) that the solution φ_k of

$$\begin{aligned}-\Delta\varphi_k &= \{a_k, b\} \quad \text{in} \quad \Omega, \\ \varphi_k &= 0 \quad \text{on} \quad \partial\Omega,\end{aligned}$$

is bounded in $W^{1,2}(\Omega)$ and therefore we have that $\varphi_k \to \eta$ weakly in $W^{1,2}(\Omega)$ with

$$||\nabla\eta||_{L^2} \leq c||\nabla a||_{L^{2,\infty}}||\nabla b||_{L^2}.$$

Additionally, since $\{a_k, b\} = \partial_x(a_k\partial_y b) - \partial_y(a_k\partial_x b)$, we have that

$$\{a_k, b\} \to \{a, b\}$$

in $W^{-1,p}$ for every $1 \leq p < 2$ and therefore

$$\varphi_k \to \varphi$$

in $W^{1,p}$ for every $1 \leq p < 2$. This shows that $\varphi = \eta$ and finishes the proof of the Theorem. □

Chapter 6

Regularity of geometric variational problems

In this chapter we study the regularity of solutions of the two-dimensional geometric variational problems which were introduced in chapter 2. More precisely we present a new result of T. Rivière [27] in which he established the continuity of the above mentioned solutions.

6.1 Gauge transformation

The main goal of this section is to give a self-contained (modulo elliptic L^p-theory) proof of the gauge transformation result of K. Uhlenbeck [37]. This result, which was first used to show the regularity of Yang-Mills fields in 4 dimensions, turned out to be very useful in the regularity theory for geometric variational problems. Good references for this section are [37], [27] and [39].

Theorem 6.1.1. *There exists $\varepsilon > 0$ and $c > 0$ such that for every $\Omega \in L^2(B^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ satisfying*

$$\int_{B^2} |\Omega|^2 < \varepsilon, \tag{6.1}$$

there exists $\xi \in W^{1,2}(B^2, so(m))$ and $P \in W^{1,2}(B^2, SO(m))$ such that

$$\begin{aligned} \nabla^\perp \xi &= P^{-1}\nabla P + P^{-1}\Omega P \quad \text{in} \;\; B^2, \\ \xi &= 0 \quad \text{on} \;\; \partial B^2, \\ \|\nabla P\|_{L^2(B^2)} + \|\xi\|_{W^{1,2}(B^2)} &\le c\|\Omega\|_{L^2(B^2)}. \end{aligned} \tag{6.2}$$

The Theorem will be a consequence of the following Lemma.

Lemma 6.1.2. *Let $1 < p < 2$. There exists $\varepsilon > 0$ and $c > 0$ such that for every $\Omega \in W^{1,p}(B^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ satisfying*

$$\int_{B^2} |\Omega|^2 < \varepsilon, \tag{6.3}$$

there exists $\xi \in W^{2,p}(B^2, so(m))$ *and* $U \in W^{2,p}(B^2, so(m))$ *such that for* $P = e^U$

$$\nabla^{\perp}\xi = P^{-1}\nabla P + P^{-1}\Omega P \quad in \quad B^2, \tag{6.4}$$

$$\xi = 0 \quad on \quad \partial B^2, \tag{6.5}$$

$$||U||_{W^{1,2}(B^2)} + ||\xi||_{W^{1,2}(B^2)} \leq c||\Omega||_{L^2(B^2)} \quad and \tag{6.6}$$

$$||U||_{W^{2,p}(B^2)} + ||\xi||_{W^{2,p}(B^2)} \leq c||\Omega||_{W^{1,p}(B^2)}. \tag{6.7}$$

Proof of Theorem 6.1.1

Let $\Omega \in L^2(B^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ satisfying (6.1) and choose $\Omega_k \in W^{1,p}(B^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ such that $\Omega_k \to \Omega$ in L^2. Then we know from Lemma 6.1.2 that there exist $\xi_k \in W^{2,p}(B^2, so(m))$ and $U_k \in W^{2,p}(B^2, so(m))$ which satisfy (6.4)-(6.7) with Ω replaced by Ω_k and $P_k = e^{U_k}$. Because of (6.6) we know that $P_k \rightharpoonup P$ weakly in $W^{1,2}$. Since weak convergence in $W^{1,2}$ implies convergence almost everywhere we conclude that $P \in W^{1,2}(B^2, SO(m))$. Moreover (up to a subsequence) P_k converges strongly in L^p for every $p < \infty$ and $P_k^{-1}\nabla P_k$ and $P_k^{-1}\Omega_k P_k$ converge to $P^{-1}\nabla P$, respecively $P^{-1}\Omega P$ in the distribution sense. Hence the first equation of (6.2) is satisfied. The second one is satisfied because of the continuity of the trace and the third one because of the lower-semicontinuity of the $W^{1,2}$-norm.

□

Proof of Lemma 6.1.2

We introduce the set

$$\mathcal{U}_{\varepsilon,C} := \{\Omega \in W^{1,p}(B^2, so(m) \otimes \wedge^1\mathbb{R}^2) | \int_{B^2} |\Omega|^2 < \varepsilon$$
$$\text{and there exist } P \text{ and } \xi \text{ satisfying} \quad (6.4) - (6.7)\}.$$

Since $\Omega \equiv 0 \in \mathcal{U}_{\varepsilon,C}$ it remains to show that, for $\varepsilon > 0$ small enough and C large enough, the set $\mathcal{U}_{\varepsilon,C}$ is both open and closed in the path-connected set

$$\mathcal{V}_\varepsilon := \{\Omega \in W^{1,p}(B^2, so(m) \otimes \wedge^1\mathbb{R}^2) | \int_{B^2} |\Omega|^2 < \varepsilon\}.$$

The set $\mathcal{V}_\varepsilon$ is path connected because for every $\Omega \in \mathcal{V}_\varepsilon$ we consider the path $\Omega_t = \varphi_t^\star\Omega$, where $\varphi_t(x) = tx$ and $t \in [0,1]$. Since $\int_{B^2} |\Omega_t|^2 = \int_{B_t^2} |\Omega|^2$ is an increasing function of t we therefore have a continuous path in $\mathcal{V}_\varepsilon$ connecting 0 and Ω.

Next we note that the closedness of the set $\mathcal{U}_{\varepsilon,C}$ follows along the same lines as the proof of Theorem 6.1.1.

Therefore it remains to show that $\mathcal{U}_{\varepsilon,C}$ is open. For this we let $\Omega \in \mathcal{U}_{\varepsilon,C}$ with the corresponding forms P and ξ.

Claim:

For every $\alpha > 0$ there exists $\delta > 0$ such that for every $\lambda \in W^{1,p}(B^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ with $||\lambda||_{W^{1,p}} \leq \delta$ there exists $\xi_\lambda \in W^{2,p}(B^2, so(m))$ and $U_\lambda \in W^{2,p}(B^2, so(m))$ such that ($P_\lambda = e^{U_\lambda}$)

$$\nabla^{\perp}\xi_\lambda = P_\lambda^{-1}\nabla P_\lambda + P_\lambda^{-1}(\nabla^{\perp}\xi + \lambda)P_\lambda \quad \text{in} \quad B^2,$$
$$\xi_\lambda = 0 \quad \text{on} \quad \partial B^2 \tag{6.8}$$

and

$$||U_\lambda||_{W^{2,p}} + ||\xi_\lambda - \xi||_{W^{2,p}} \le \alpha. \tag{6.9}$$

If we accept this claim for a second we can continue with the proof of the Lemma. Namely, (6.4) and (6.8) imply that

$$\nabla^\perp \xi_\lambda = (PP_\lambda)^{-1}\nabla(PP_\lambda) + (PP_\lambda)^{-1}(\Omega + P\lambda P^{-1})PP_\lambda. \tag{6.10}$$

Since $\nabla P \in W^{1,p}(B^2, SO(m)) \hookrightarrow L^{\frac{2p}{2-p}}(B^2, SO(m))$ the map $\lambda \to P\lambda P^{-1}$ and its inverse $\zeta \to P^{-1}\zeta P$ are continuous from $W^{1,p}(B^2, so(m))$ into itself (for example

$$\begin{aligned}||\nabla(P\lambda P^{-1})||_{L^p} \le& c||\lambda||_{W^{1,p}} + c||\lambda\nabla P||_{L^p}\\ \le& c||\lambda||_{W^{1,p}} + c||\lambda||_{L^{\frac{2p}{2-p}}}||\nabla P||_{L^2}\\ \le& c||\lambda||_{W^{1,p}}(1 + ||\nabla P||_{L^2})).\end{aligned}$$

Therefore, for every $\beta > 0$ there exists $\eta > 0$ such that for every $\omega \in W^{1,p}(B^2, so(m) \otimes \wedge\Lambda^1\mathbb{R}^2)$ with $||\omega||_{W^{1,p}} \le \eta$ there exists $S \in W^{2,p}(B^2, so(m))$ and $\nu \in W^{2,p}(B^2, so(m))$ such that $(R = e^S)$

$$\begin{aligned}\nabla^\perp \nu &= R^{-1}\nabla R + R^{-1}(\Omega + \omega)R \quad \text{in} \quad B^2,\\ \nu &= 0 \quad \text{on} \quad \partial B^2.\end{aligned} \tag{6.11}$$

Moreover we have that

$$||S - U||_{W^{2,p}} + ||\nu - \xi||_{W^{2,p}} < \beta. \tag{6.12}$$

To see this we use the above continuity argument and choose $S = U + U_\lambda$ and $\nu = \xi_\lambda$, the the result follows from (6.9) and (6.10). Now we consider $\beta < \sqrt{\varepsilon}$ and get (since by (6.6) $||\nabla P||_{L^2} + ||\xi||_{W^{1,2}} \le c\sqrt{\varepsilon}$)

$$||\nabla R||_{L^2} + ||\nu||_{W^{1,2}} \le (c+1)\sqrt{\varepsilon}. \tag{6.13}$$

In order to complete the proof of the openness of $U_{\varepsilon,C}$ it remains to show that R and ν satisfy (6.6) and (6.7). This follows from the following

Lemma 6.1.3. *There exists $c > 0$ and $\delta > 0$ such that for every $P \in W^{2,p}(B^2, SO(m))$ and $\xi \in W^{2,p}(B^2, so(m))$ satisfying (6.4) and (6.5) for some $\Omega \in W^{1,p}(B^2, so(m) \otimes \wedge^1\mathbb{R}^2)$ with $\int_{B^2} |\Omega|^2 \le \varepsilon$ and*

$$||\nabla P||_{L^2} + ||\xi||_{W^{1,2}} \le \delta, \tag{6.14}$$

then (6.6) and (6.7) are satisfied.

Proof. Because of (6.4) and (6.5) we see that ξ solves

$$\begin{aligned}\Delta\xi &= \nabla^\perp P^{-1}\nabla P + \operatorname{div}(P^{-1}\Omega P) \quad \text{in} \quad B^2,\\ \xi &= 0 \quad \text{on} \quad \partial B^2.\end{aligned} \tag{6.15}$$

Using Wente's inequality (Theorem 3.2.1) and standard L^2-theory we conclude

$$||\nabla\xi||_{L^2} \leq c||\nabla P||_{L^2}^2 + c||\Omega||_{L^2}. \tag{6.16}$$

Multiplying (6.4) with P from the left we get

$$||\nabla P||_{L^2} \leq c||\nabla\xi||_{L^2} + c||\Omega||_{L^2}. \tag{6.17}$$

Inserting (6.17) into (6.16) and using that $||\nabla P||_{L^2} \leq \delta$ we get

$$||\nabla\xi||_{L^2} \leq c\delta||\nabla\xi||_{L^2} + c(1+\delta)||\Omega||_{L^2}. \tag{6.18}$$

Choosing δ small enough we get the estimate (6.6) by combining (6.18) and (6.17).
Now it remains to prove (6.7). We consider (6.15) and use standard elliptic L^p-theory to get (we use that $||fg||_{L^p} \leq c||f||_{L^2}||g||_{W^{1,p}}$)

$$\begin{aligned}||\xi||_{W^{2,p}} \leq& c||\nabla^\perp P^{-1}\nabla P||_{L^p} + c||\Omega||_{W^{1,p}} + c||\nabla P\Omega||_{L^p}\\ \leq& c||\nabla P||_{L^2}||P||_{W^{2,p}} + c||\Omega||_{W^{1,p}} + c||P||_{W^{2,p}}||\Omega||_{L^2} + c||\nabla P||_{L^2}||\Omega||_{W^{1,p}}\\ \leq& c(\delta+\sqrt{\varepsilon})||P||_{W^{2,p}} + c||\Omega||_{W^{1,p}}.\end{aligned} \tag{6.19}$$

Using (6.4) we get

$$\begin{aligned}||\nabla P||_{W^{1,p}} \leq& c||\xi||_{W^{2,p}} + ||\nabla\xi\nabla P||_{L^p} + ||\nabla(P^{-1}\Omega P)||_{L^p}\\ \leq& c(1+||\nabla P||_{L^2})||\xi||_{W^{2,p}} + c||\nabla\xi||_{L^2}||\nabla P||_{W^{1,p}} + c||\Omega||_{W^{1,p}} + c||\nabla P\Omega||_{L^p}\\ \leq& c(1+\delta)||\xi||_{W^{2,p}} + c(\delta+\sqrt{\varepsilon})||\nabla P||_{W^{1,p}} + c(1+\delta)||\Omega||_{W^{1,p}}.\end{aligned} \tag{6.20}$$

Choosing δ small enough and combining (6.19) and (6.20) we get (6.7). □

It remains to prove the claim.
Proof of the claim:
We will prove the claim by an application of the implicit function theorem. In order to do this we define the function spaces

$$\begin{aligned}W^{2,p}_m(B^2, so(m)) &= \{f \in W^{2,p}(B^2, so(m)) | \int_{B^2} f = 0\}\\ W^{1,p}_\partial(B^2, so(m)) &= \{g|_{\partial B^2} | g \in W^{1,p}(B^2, so(m))\},\end{aligned}$$

where we define the norm of $W^{1,p}_\partial(B^2, so(m))$ by

$$||g||_{W^{1,p}_\partial} = \inf\{||G||_{W^{1,p}} | G \in W^{1,p}(B^2, so(m)), G|_{\partial B^2} = g\}.$$

It is easy to check that both spaces are Banach spaces. Next we define the operator

$$\begin{aligned}D :& W^{1,p}(B^2, so(m)) \times W^{2,p}_m(B^2, so(m)) \to Z\\ D(\lambda, U) =& (\operatorname{div}(e^{-U}\nabla e^U + e^{-U}(\nabla^\perp\xi+\lambda)e^U), (e^{-U}\nabla e^U + e^{-U}(\nabla^\perp\xi+\lambda)e^U)\cdot\nu),\end{aligned} \tag{6.21}$$

where ν is the unit normal to ∂B^2 and

$$Z = \{(f,g) \in L^p(B^2, so(m)) \times W^{1,p}_\partial(B^2, so(m)) | \int_{B^2} f = \int_{\partial B^2} g\}.$$

Again it is easy to check (using Stokes Theorem, Sobolev embedding and the properties of U and λ) that D is well-defined and that (remember $\xi|_{\partial B^2} = 0$)

$$D(0,0) = 0. \tag{6.22}$$

Next we have to calculate the second derivative (i.e. the linearization) of D with respect to the second component. To do so we calculate for $Q \in W^{2,p}_m(B^2, so(m))$ (using $\xi = 0$ on ∂B^2)

$$\frac{\partial D(0, \varepsilon Q)}{\partial \varepsilon}|_{\varepsilon=0} = D_2D(0,0)(Q) = (\Delta Q + \nabla^\perp \xi \nabla Q - \nabla Q \nabla^\perp \xi, \frac{\partial Q}{\partial \nu}), \tag{6.23}$$

where $D_2D(0,0) : W^{2,p}_m(B^2, so(m)) \to Z$. Now we have to show that $D_2D(0,0)$ is bijective. To do this we decompse $D_2D(0,0) = T + S$, where

$$\begin{aligned} T(Q) &= (\Delta Q, \frac{\partial Q}{\partial \nu}), \\ S(Q) &= (\nabla^\perp \xi \nabla Q - \nabla Q \nabla^\perp \xi, 0). \end{aligned}$$

From standard elliptic L^p-theory (see for example [39], chapter 3) it is well known that the operator $T : W^{2,p}_m(B^2, so(m)) \to Z$ is bijective (here one essentially has to use that the elements in the domain have mean value zero and the integral assumption in the definition of Z) and that moreover for every $Q \in W^{2,p}_m(B^2, so(m))$

$$||Q||_{W^{2,p}} \le c_1(||\Delta Q||_{L^p} + ||\frac{\partial Q}{\partial \nu}||_{W^{1,p}_\partial}) = c_1||T(Q)||_Z. \tag{6.24}$$

This estimate directly gives an estimate for the inverse operator $T^{-1} : Z \to W^{2,p}_m(B^2, so(m))$, namely

$$||T^{-1}|| \le c_1. \tag{6.25}$$

Next we claim that for every $Q \in W^{2,p}(B^2, so(m))$

$$||S(Q)||_Z \le c\sqrt{\varepsilon}||Q||_{W^{2,p}}. \tag{6.26}$$

Namely we have

$$\begin{aligned} ||S(Q)||_Z =& ||\nabla^\perp \xi \nabla Q - \nabla Q \nabla^\perp \xi||_{L^p} \\ \le& c||\nabla \xi||_{L^2}||\nabla^2 Q||_{L^p}. \end{aligned}$$

Hence we can apply the following

Lemma 6.1.4. *Let $T, T^{-1}, S : X \to Z$ be bounded linear operators between Banach spaces. Suppose that T is bijective and that $||T^{-1}||||S|| < 1$. Then the operator $T + S$ is also bijective and*

$$||(T+S)^{-1}|| \le \frac{||T^{-1}||}{1 - ||T^{-1}||||S||} \tag{6.27}$$

$$||(T+S)^{-1} - T^{-1}|| \le \frac{||T^{-1}||^2||S||}{1 - ||T^{-1}||||S||}. \tag{6.28}$$

Proof. We have

$$(T+S)^{-1} = \sum_{k=0}^{\infty}(-T^{-1}S)^k T^{-1} = \sum_{k=0}^{\infty} T^{-1}(-ST^{-1})^k.$$

This follows because $||T^{-1}S|| \le ||T^{-1}||||S|| < 1$ and hence the series converges and is the left and right inverse of $T+S$:

$$(\sum_{k=0}^{N}(-T^{-1}S)^k T^{-1})(T+S) = id - (-T^{-1}S)^{N+1} \to id$$
$$(T+S)(\sum_{k=0}^{N} T^{-1}(-ST^{-1})^k) = id - (-ST^{-1})^{N+1} \to id.$$

Moreover this also gives

$$||(T+S)^{-1}|| \le \sum_{k=0}^{\infty} ||T^{-1}||^{k+1}||S||^k = \frac{||T^{-1}||}{1-||T^{-1}||||S||}$$

and

$$\begin{aligned} ||(T+S)^{-1} - T^{-1}|| &\le ||(T+S)^{-1}||||T-(T+S)||||T^{-1}|| \\ &\le \frac{||T^{-1}||^2||S||}{1-||T^{-1}||||S||}. \end{aligned}$$

□

Altogether this shows that $D_2D(0,0)$ is bijective and hence we can apply the implicit function theorem to get that for every $\lambda \in W^{1,p}(B^2, so(m))$ with $||\lambda||_{W^{1,p}} \le \delta$ there exists $U_\lambda \in W^{2,p}(B^2, so(m))$ solving

$$D(\lambda, U_\lambda) = 0$$

and

$$||U_\lambda||_{W^{2,p}} \le \alpha.$$

Defining $P_\lambda = e^{U_\lambda}$ we get that

$$||P_\lambda - id_m||_{W^{2,p}} \le \alpha.$$

Since now $\mathrm{div}(P_\lambda^{-1}\nabla P_\lambda + P_\lambda^{-1}(\nabla^\perp\xi + \lambda)P_\lambda) = 0$ and $(P_\lambda^{-1}\nabla P_\lambda + P_\lambda^{-1}(\nabla^\perp\xi + \lambda)P_\lambda) \cdot \nu = 0$ we can apply Lemma 3.3.1 (which obviously also holds in L^p-spaces) to get the existence of a map $\xi_\lambda \in W^{2,p}(B^2, so(m))$ such that

$$\begin{aligned} \nabla^\perp\xi_\lambda &= P_\lambda^{-1}\nabla P_\lambda + P_\lambda^{-1}(\nabla^\perp\xi + \lambda)P_\lambda \quad \text{in} \quad B^2, \\ \xi_\lambda &= 0 \quad \text{on} \quad \partial B^2 \end{aligned}$$

and (using again that $||fg||_{L^p} \le c||f||_{L^2}||g||_{W^{1,p}}$)

$$||\xi_\lambda||_{W^{2,p}} \le \alpha.$$

This proves the claim and hence the Lemma. □

6.2 Equations of the form $\Delta u = \Omega\nabla u$

Let $B \subset \mathbb{R}^2$ be the unit ball in $\mathbb{R}^2$. In this section we study the regularity properties of solutions of elliptic systems of the form

$$-\Delta u = \Omega\nabla u, \tag{6.29}$$

where $u \in W^{1,2}(B, \mathbb{R}^m)$ and $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$. Before coming to the detailed study let us give some examples for systems of the type (6.29).

1) From (3.41) we see that harmonic maps into spheres satisfy an equation of the form (6.29) with $(\Omega^{ij}) = (u^i\nabla u^j - u^j\nabla u^i) \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$.

2) It is easy to see that surfaces with prescribed mean curvature $H \in L^\infty(\mathbb{R}^3)$ (i.e. solutions of (2.7)) solve a system of the form (6.29) with

$$\Omega = -2H(u)\begin{pmatrix} 0 & \nabla^\perp u^3 & -\nabla^\perp u^2 \\ -\nabla^\perp u^3 & 0 & \nabla^\perp u^1 \\ \nabla^\perp u^2 & -\nabla^\perp u^1 & 0 \end{pmatrix} \in L^2(B, so(3) \otimes \wedge^1\mathbb{R}^2).$$

3) Harmonic maps into general target manifolds.
Here we let $u \in W^{1,2}(B, N)$, where $N \hookrightarrow \mathbb{R}^m$ is a smooth and compact Riemannian manifold without boundary. Then we know from the discussions in chapter 2 that harmonic maps into N are critical points of the functional

$$E(u) = \frac{1}{2}\int_B |\nabla u|^2 dv_g.$$

To compute the critical points of E we let $\varphi \in C_c^1(B, \mathbb{R}^m)$ with $\varphi(x) \in T_{u(x)}N$ for all $x \in B$. Then we compute

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} E(u + t\varphi) \\ &= -\int_B \Delta u \varphi. \end{aligned}$$

Since this is true for all such φ we know that

$$\Delta u \perp T_u N.$$

Therefore if we let $\{\nu_{n+1}, \ldots, \nu_m\}$ be a smooth local orthonormal frame for the normal bundle near $u(x)$ we can write

$$\Delta u(x) = \sum_{i=n+1}^{m} \lambda_i(x)\nu_i(u(x)),$$

where the λ_i are scalar functions. Using the fact that $\langle \nabla u, \nu_i(u)\rangle = 0$ for every $i \in \{n+1, \ldots, m\}$ we get

$$\begin{aligned} \lambda_i &= \langle \Delta u, \nu_i(u)\rangle \\ &= \operatorname{div}\langle \nabla u, \nu_i(u)\rangle - \langle \nabla_j u, (d_k\nu_i)(u)\nabla_j u^k\rangle \end{aligned}$$

and hence

$$\begin{aligned}\Delta u =& \sum_i \lambda_i \nu_i(u)\\ =& -\sum_{i=n+1}^{m}\sum_{k=1}^{m}\sum_{j=1}^{2}\langle \nabla_j u, (d_k\nu_i)(u)\nabla_j u^k\rangle \nu_i(u)\\ =& -A(u)(\nabla u, \nabla u).\end{aligned}$$

Moreover, using the definition of A, we see that (using that $\sum_k \nabla u^k \nu_i^k(u) = 0$ for every i)

$$\begin{aligned}\Delta u^s =& -\sum_{i,k}\langle \nabla u, (d_k\nu_i)(u)\nabla u^k\rangle \nu_i^s(u)\\ =& -\sum_{i,k,l}\nabla u^k(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l),\end{aligned}$$

and hence u solves an equation of the form (6.29) with

$$(\Omega_{sk}) = (\sum_{i,l}(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l)) \in L^2(B, so(m)\otimes \wedge^1\mathbb{R}^2).$$

4) Conformally invariant variational problems.
We consider the functional

$$E_\omega(u) = \frac{1}{2}\int_B (|\nabla u|^2 + \omega(u)(\partial_x u, \partial_y u))dx,$$

where ω is a C^1 two-form on $\mathbb{R}^m$ such that the L^∞-norm of $d\omega$ is bounded. By Theorem 2.3.1 we see that every conformally invariant energy in two-dimensions can be written in this way. The Euler-Lagrange equation of E_ω can easily be computed to be

$$\Delta u^i + A^i(u)(\nabla u, \nabla u) + \lambda^i_{jl}(u)\partial_x u^j \partial_y^l = 0,$$

where $\lambda^i_{jl}(u) = d\omega(u)(e_i, e_j, e_l)$ and where $\{e_i\}_{\{i=1,\dots,m\}}$ is the standard basis of $\mathbb{R}^m$. Using that $\lambda^i_{jl} = -\lambda^j_{il}$ we calculate

$$\lambda^i_{jl}(u)\partial_x u^j \partial_y^l = \frac{1}{4}(\lambda^i_{jl}(u) - \lambda^j_{il}(u))\nabla^\perp u^l \nabla u^j.$$

Combining this with the result of 3) we see that the Euler-Lagrange equation of every conformally invariant energy in two dimensions can be written in the form (6.29) with

$$\begin{aligned}\Omega_{sk} =& \sum_{i,l}(\nu_i^s(u)(d_k\nu_i)^l(u)\nabla u^l - \nu_i^k(u)(d_s\nu_i)^l(u)\nabla u^l)\\ &-\sum_l \frac{1}{4}(\lambda^s_{kl}(u) - \lambda^k_{sl}(u))\nabla^\perp u^l\\ &\in L^2(B, so(m)\otimes \wedge^1\mathbb{R}^2).\end{aligned}$$

After having collected all these examples of systems of the type (6.29) we now state the main Theorem of this chapter. This Theorem was only recently proved by Tristan Rivière [27] (see also [21], [28] and [36] for related results).

Theorem 6.2.1. *Let $u \in W^{1,2}(B,\mathbb{R}^m)$ be a solution of (6.29) with $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$. Then u is continuous and therefore by Theorem 3.4.1 as smooth as the data permits.*

Proof. The Theorem will be proved in three steps.
Step 1:

Lemma 6.2.2. *Let $m \in \mathbb{N}$ and $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$. Let $A \in L^\infty \cap W^{1,2}(B, M(m))$ and $B \in W^{1,2}(B, M(m))$ be solutions of*

$$\nabla A - A\Omega = \nabla^\perp B. \tag{6.30}$$

Then $u \in W^{1,2}(B,\mathbb{R}^m)$ is a solution of (6.29) with Ω iff

$$\operatorname{div}(A\nabla u + B\nabla^\perp u) = 0. \tag{6.31}$$

Proof. By a direct calculation (using that $\operatorname{div}\nabla^\perp = 0$ and $\nabla u \nabla^\perp v = -\nabla^\perp u \nabla v$) and using (6.30) we get

$$\begin{aligned}\operatorname{div}(A\nabla u + B\nabla^\perp u) =& (\nabla A - \nabla^\perp B)\nabla u + A\Delta u \\ =& A(\Delta u + \Omega\nabla u).\end{aligned}$$

This proves the Lemma. □

Step 2:

Lemma 6.2.3. *There exists $\varepsilon > 0$, $c > 0$ such that for every $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$ with*

$$\int_B |\Omega|^2 dx < \varepsilon, \tag{6.32}$$

there exist $A \in L^\infty \cap W^{1,2}(B, Gl(m))$ and $B \in W^{1,2}(B, M(m))$ satisfying

$$\int_B (|\nabla A|^2 + |\nabla B|^2)dx + ||\operatorname{dist}(A, SO(n))||^2_{L^\infty} \le c\int_B |\Omega|^2 \quad \textit{and} \tag{6.33}$$

$$\nabla A - A\Omega - \nabla^\perp B = 0. \tag{6.34}$$

Proof. For $\Omega \in L^2(B, so(m) \otimes \wedge^1\mathbb{R}^2)$ with $\int_B |\Omega|^2 dx < \varepsilon$ we apply Theorem 6.1.1 to get the existence of $P \in W^{1,2}(B, SO(m))$ and $\xi \in W^{1,2}(B, so(m))$ such that $\xi = 0$ on ∂B,

$$\nabla^\perp \xi = P^{-1}\nabla P + P^{-1}\Omega P. \tag{6.35}$$

and

$$||\xi||_{W^{1,2}} + ||\nabla P||_{L^2} + ||\nabla P^{-1}||_{L^2} \le c||\Omega||_{L^2}. \tag{6.36}$$

Claim 1: There exist $\hat{A} \in W^{1,2} \cap L^\infty(B, M(m))$ and $B \in W^{1,2}(B, M(m))$ solving

$$\Delta \hat{A} = \nabla \hat{A} \nabla^\perp \xi + \nabla^\perp B \nabla P \quad \text{in} \quad B, \tag{6.37}$$

$$\Delta B = -\nabla^\perp \hat{A} \nabla P^{-1} - \operatorname{div}(\hat{A} \nabla \xi P^{-1} + \nabla \xi P^{-1}) \quad \text{in} \quad B, \tag{6.38}$$

$$\frac{\partial \hat{A}}{\partial \nu} = 0 \quad \text{and} \quad B = 0 \quad \text{on} \quad \partial B, \tag{6.39}$$

$$\int_B \hat{A} = 0. \tag{6.40}$$

To prove this claim we apply Theorem 3.2.1 (combined with remark 3.2.4) and standard L^2-theory to get

$$||\hat{A}||_{W^{1,2}} + ||\hat{A}||_{L^\infty} \leq c||\nabla \xi||_{L^2}||\nabla \hat{A}||_{L^2} + c||\nabla P||_{L^2}||\nabla B||_{L^2} \quad \text{and} \tag{6.41}$$

$$||B||_{W^{1,2}} \leq c||\nabla P^{-1}||_{L^2}||\nabla \hat{A}||_{L^2} + c||\nabla \xi||_{L^2}||\hat{A}||_{L^\infty} + c||\nabla \xi||_{L^2}. \tag{6.42}$$

Using (6.36) and choosing ε small enough we combine (6.41) and (6.42) to get

$$||\hat{A}||_{W^{1,2}} + ||\hat{A}||_{L^\infty} + ||B||_{W^{1,2}} \leq c||\Omega||_{L^2}. \tag{6.43}$$

The existence of the desired solution of (6.37)-(6.40) (and hence the proof of Claim 1) now follows from a standard fixed-point argument.
Next we define $\tilde{A} = \hat{A} + id$ and we see from (6.37)-(6.40) that $\tilde{A}$ and B solve

$$\Delta \tilde{A} = \nabla \tilde{A} \nabla^\perp \xi + \nabla^\perp B \nabla P \quad \text{in} \quad B, \tag{6.44}$$

$$\Delta B = -\nabla^\perp \tilde{A} \nabla P^{-1} - \operatorname{div}(\tilde{A} \nabla \xi P^{-1}) \quad \text{in} \quad B, \tag{6.45}$$

$$\frac{\partial \tilde{A}}{\partial \nu} = 0 \quad \text{and} \quad B = 0 \quad \text{on} \quad \partial B, \tag{6.46}$$

$$\int_B \tilde{A} = |B|. \tag{6.47}$$

Moreover we get from (6.43) that

$$||\nabla \tilde{A}||_{L^2} + ||\operatorname{dist}(\tilde{A}, SO(m))||_{L^\infty} + ||B||_{W^{1,2}} \leq c||\Omega||_{L^2}. \tag{6.48}$$

Now it is easy to see that (6.44) can be rewritten as

$$\operatorname{div}(\nabla \tilde{A} - \tilde{A} \nabla^\perp \xi - \nabla^\perp B P) = 0 \tag{6.49}$$

and hence, by Lemma 3.3.1, there exists $C \in W^{1,2}(B, M(m) \otimes \wedge^1 \mathbb{R}^2)$ such that

$$\nabla \tilde{A} - \tilde{A} \nabla^\perp \xi - \nabla^\perp B P = \nabla^\perp C. \tag{6.50}$$

Since by (6.46) and the definition of ξ we have

$$(\nabla \tilde{A} - \tilde{A} \nabla^\perp \xi - \nabla^\perp B P) \cdot \nu = \frac{\partial \tilde{A}}{\partial \nu} - \tilde{A} \nabla^\perp \xi \cdot \nu - \nabla^\perp B P \cdot \nu$$
$$= 0$$

on ∂B we can moreover assume that $C = 0$ on ∂B. Using a rotation by $\frac{\pi}{2}$ (one can also view $\nabla^\perp$ as $\star d$ and then the rotation by $\frac{\pi}{2}$ is just another application of $\star$) we see that (6.50) is equivalent to

$$-\nabla CP^{-1} = \nabla^\perp \tilde{A}P^{-1} + \tilde{A}\nabla\xi P^{-1} + \nabla B,$$

and hence, using (6.45), we calculate

$$\begin{aligned} -\operatorname{div}(\nabla CP^{-1}) &= \nabla^\perp \tilde{A}\nabla P^{-1} + \operatorname{div}(\tilde{A}\nabla\xi P^{-1}) + \Delta B \\ &= 0. \end{aligned} \tag{6.51}$$

<u>Claim 2:</u> Every solution C of (6.51) with $C = 0$ on ∂B vanishes identically.
To see this we apply again Lemma 3.3.1 and get the existence of $D \in W^{1,2}(B, M(m) \otimes \wedge^1\mathbb{R}^2)$ such that

$$\nabla^\perp D = \nabla CP^{-1}. \tag{6.52}$$

Since $C = 0$ on ∂B we easily see that $\frac{\partial D}{\partial \nu} = 0$ on ∂B and we can also assume that $\int_B D = 0$. Hence C and D solve

$$\Delta C = \nabla^\perp D\nabla P \quad \text{in} \quad B, \tag{6.53}$$

$$\Delta D = \nabla C\nabla^\perp P^{-1} \quad \text{in} \quad B, \tag{6.54}$$

$$C = 0 \quad \text{and} \quad \frac{\partial D}{\partial \nu} = 0 \quad \text{on} \quad \partial B, \tag{6.55}$$

$$\int_B D = 0. \tag{6.56}$$

From this we see that we can apply Theorem 3.2.1 for (6.53) and (6.54) (in this case combined with remark 3.2.4) to get

$$||\nabla C||_{L^2} + ||\nabla D||_{L^2} \le c(||\nabla P||_{L^2}||\nabla D||_{L^2} + ||\nabla P^{-1}||_{L^2}||\nabla C||_{L^2}). \tag{6.57}$$

By choosing ε small enough we get from (6.36) that $C = D = 0$ and this shows the claim.
From (6.50) we now see that $\tilde{A}$ and B solve

$$\nabla\tilde{A} - \tilde{A}\nabla^\perp\xi - \nabla^\perp BP = 0. \tag{6.58}$$

Defining $A = \tilde{A}P^{-1}$ we see that

$$\begin{aligned} ||\nabla A||_{L^2} + ||\operatorname{dist}(A, SO(m))||_{L^\infty} &\le c(||\nabla\tilde{A}||_{L^2} + ||\tilde{A}||_{L^\infty}||\nabla P^{-1}||_{L^2} + ||\operatorname{dist}(\tilde{A}, SO(m))||_{L^\infty}) \\ &\le c||\Omega||_{L^2}, \end{aligned} \tag{6.59}$$

where we used (6.36) and (6.48). Moreover we use (6.35) and (6.58) to calculate

$$\begin{aligned} 0 = \nabla\tilde{A} - \tilde{A}\nabla^\perp\xi - \nabla^\perp BP &= \nabla AP + A\nabla P - AP\nabla^\perp\xi - \nabla^\perp BP \\ &= A\nabla P - A\nabla P + (\nabla A - A\Omega - \nabla^\perp B)P \end{aligned}$$

and therefore

$$\nabla A - A\Omega - \nabla^\perp B = 0. \tag{6.60}$$

This finishes the proof of the Lemma. □

Step 3:
For every point $x \in B$ we choose a radius $r_x > 0$ such that $\int_{B_{r_x}(x)} |\Omega|^2 < \varepsilon$, where ε is the same as in Lemma 6.2.3. In the following we write $B_{r_x}(x) = B$. Then we can apply Lemma 6.2.3 to get the existence of A and B solving (6.30). Hence we can apply Lemma 6.2.2 to see that

$$\operatorname{div}(A\nabla u) = \nabla B \nabla^{\perp} u = -\nabla^{\perp} B \nabla u \quad \text{and} \tag{6.61}$$

$$\nabla^{\perp}(A\nabla u) = \nabla^{\perp} A \nabla u. \tag{6.62}$$

Now we apply Lemma 3.3.1 to get the existence of $\alpha \in W^{1,2}(B, \mathbb{R}^m)$ and $\beta \in W^{1,2}(B, \mathbb{R}^m \otimes \wedge^1 \mathbb{R}^2)$ such that

$$A\nabla u = \nabla \alpha + \nabla^{\perp} \beta. \tag{6.63}$$

Using (6.61) we see that α solves

$$\Delta \alpha = \operatorname{div}(A\nabla u) = -\nabla^{\perp} B \nabla u. \tag{6.64}$$

Now we denote by $\overline{u}$ the mean value of u on $B_{\frac{1}{2}}$ and let $\tilde{u} \in W_0^{1,2}(\mathbb{R}^2, \mathbb{R}^m)$ be the extension with compact support of $u - \overline{u}$. Then we have that $\nabla \tilde{u} = \nabla u$ on $B_{\frac{1}{2}}$. Moreover we use Poincare's inequality to get

$$||\nabla \tilde{u}||_{L^2(\mathbb{R}^2)} \leq c||u - \overline{u}||_{W^{1,2}(B_{\frac{1}{2}})} \leq c||\nabla u||_{L^2(B)}.$$

We extend B in the same way and denote the resulting map by $\tilde{B} \in W_0^{1,2}(\mathbb{R}^2, M(m))$. Then we let $\tilde{\alpha}$ be the solution of

$$\Delta \tilde{\alpha} = -\nabla^{\perp} \tilde{B} \nabla \tilde{u} \tag{6.65}$$

on $\mathbb{R}^2$. Since by Corollary 4.2.13 $-\nabla^{\perp} \tilde{B} \nabla \tilde{u} \in \mathcal{H}^1(\mathbb{R}^2)$ with

$$|| - \nabla^{\perp} \tilde{B} \nabla \tilde{u}||_{\mathcal{H}^1(\mathbb{R}^2)} \leq c||\nabla B||_{L^2(B)} ||\nabla u||_{L^2(B)}$$

we can apply Theorem 4.3.8 to get that $\tilde{\alpha} \in W^{2,1}(\mathbb{R}^2)$. Since

$$\Delta(\alpha - \tilde{\alpha}) = 0$$

on $B_{\frac{1}{2}}$ we get that $\alpha \in W^{2,1}(B_{\frac{1}{4}})$ (harmonic functions are smooth). Next we observe that β solves

$$\Delta \beta = \nabla^{\perp} A \nabla u \tag{6.66}$$

and hence we can argue as before to get that $\beta \in W^{2,1}(B_{\frac{1}{4}})$. Therefore we see from (6.63) that

$$A\nabla u \in W^{1,1}(B_{\frac{1}{4}}) \tag{6.67}$$

and therefore (using (6.33))

$$\nabla u \in W^{1,1}(B_{\frac{1}{4}}) \quad \text{or} \quad u \in W^{2,1}(B_{\frac{1}{4}}). \tag{6.68}$$

Combining this with Corollary 5.1.20 we finish the proof of the Theorem. □

With the following counterexamples of Frehse [11] we show that one can not drop the condition that Ω has to be antisymmetric.

Remark 6.2.4. *Let* $u = (u_1, u_2) \in W^{1,2}(B, S^1 \subset \mathbb{R}^2)$ *be defined by*

$$u_1(x) = \sin \ln \ln \frac{2}{|x|},$$
$$u_2(x) = \cos \ln \ln \frac{2}{|x|}$$

then it is easy to check that u *solves the elliptic system* $-\Delta u = \Omega \nabla u$ *with*

$$\Omega = \begin{pmatrix} (u_1 + u_2)\nabla u_1 & (u_1 + u_2)\nabla u_2 \\ (u_2 - u_1)\nabla u_1 & (u_2 - u_1)\nabla u_2 \end{pmatrix}.$$

So in this case Ω *is not antisymmetric and* u *is bounded but not continuous.*
For $u \in W^{1,2}(B, \mathbb{R}^2)$ *given by*

$$u_1(x) = \ln \ln \frac{2}{|x|},$$
$$u_2(x) = \ln \ln \frac{2}{|x|}$$

we have that u *solves the elliptic system* $-\Delta u = \Omega \nabla u$ *with*

$$\Omega = \begin{pmatrix} \nabla u_1 & 0 \\ 0 & \nabla u_2 \end{pmatrix}.$$

Here we even don‘t have that u *is bounded.*

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