Publications

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Refereed Papers

We consider a numerical technique to verify exact eigenvalues and eigenfunctions of second-order elliptic operators in some neighborhood of their approximations. We construct, on the computer, a set which satisfies the hypotheses of Schauder’s fixed point theorem for a compact map in a certain Sobolev space, and which therefore contains a solution. Moreover, we propose a method to bound the eigenvalue which has the smallest absolute value. A numerical example is presented.

We propose a numerical method to verify the existence and local uniqueness of solutions to nonlinear elliptic equations. We numerically construct a set which satisfies the hypotheses of Banach’s fixed point theorem in a certain Sobolev space, and which therefore contains a locally unique solution. By using finite element approximations and constructive error estimates, we calculate a lower bound for the modulus of the eigenvalue with smallest absolute value, to evaluate the norm of the inverse of the linearized operator. Utilizing this bound we derive a verification condition of Newton-Kantorovich type. Numerical examples are presented.

We propose a numerical method to enclose eigenvalues and eigenfunctions of second-order elliptic operators, proving also local uniqueness properties. We numerically construct a set containing eigenpairs which satisfies the hypotheses of Banach’s fixed point theorem in a certain Sobolev space, by using a finite element approximation and constructive error estimates. We then prove the local uniqueness separately for eigenvalues and eigenfunctions. This local uniqueness assures the simplicity of the eigenvalue. Numerical examples are presented.

We consider eigenvalue enclosing for the elliptic operator which is the linearization at an exact solution of some nonlinear elliptic equation. This problem is important in the mathematically rigorous analysis of stability or bifurcation properties of some solutions to nonlinear problems. We formulate such kind of eigenvalue problem as a nonlinear system which contains both the linearized eigenvalue problem and the original nonlinear equation. We also control the indices of
eigenvalues, and thus especially we can consider the first eigenvalue of such a problem. In these enclosing procedures, finite dimensional verified computations for linear and nonlinear systems of equations play an essential role. A numerical example is presented.


We study the eigenvalue problem for an operator which defines a sort of non-trivial coupling of usual harmonic oscillators. Since one has only a limited understanding of its eigenvalues, of the behavior of eigenfunctions, and in particular of the multiplicity of eigenvalues, here we try to make a numerical approach to this system. More precisely, applying a numerical enclosure method for elliptic eigenvalue problems which is based on a verification procedure for nonlinear elliptic equations, adapted to such coupled-type eigenvalue problems on an unbounded domain, we develop a verified numerical computation for eigenvalues which also gives guaranteed information about multiplicity.


Eigenvalues of linear operators take an important role to understand a nonlinear phenomenon in science and engineering. Especially, it often becomes a key value when we consider a behaviour of dynamical systems. This paper concerns with eigenvalue problems for matrices and differential operators. We first deal with a theoretical framework for it by introducing a concept of spectrum, and consider matrix eigenvalue problems. Then we concern with infinite dimensional eigenvalue problems to obtain eigenvalues and eigenvectors for an operator in infinite dimensional domain of definition. We explain it from classical procedure to our recent results based on computer assisted proofs. We also deal with some applications of the enclosed eigenvalues and eigenvectors to algebra, another numerical verification method for nonlinear problems and stability analysis of bifurcation phenomenon in hydrodynamics.


There exists a huge number of references concerned with bifurcations and stability results for the Navier-Stokes equations. Only a few, however, provide a rigorous result which guarantees stability or instability. Our aim is to present a rigorous theorem which proves the stability of certain solutions arising in what is called the Kolmogorov problem. We accomplish this by verified computation. The eigenvalue problem arising in the Kolmogorov problem is not self-adjoint and, accordingly, it is quite difficult to treat theoretically. Our method is a rigorous numerical approach to deal with this difficulty, and numerical examples are given as a demonstration.


In numerical verification methods for solutions of nonlinear fourth order elliptic equations in nonconvex polygonal domains, it is important to find explicitly the crucial constant in constructive a priori error estimates for the finite element approximation of bi-harmonic problems. We
construct such procedures by verified computational techniques, using Hermite spline functions, for a two dimensional L-shaped domain. Several numerical examples which confirm the actual effectiveness of the method are presented.

We present a numerical method to enclose stationary solutions of the Navier-Stokes equations, especially of the 2-D driven cavity problem with regularized boundary condition. Our method is based on an infinite dimensional Newton method, which in particular needs bounds for the inverse of the corresponding linearized operator. The method can be applied to problems with high Reynolds numbers, and we show some numerical examples which confirm the actual effectiveness.

We show how guaranteed bounds for eigenvalues (together with eigenvectors) are obtained and how non-existence of eigenvalues in a concrete region can be assured. Some examples for several types of operators in bounded and unbounded domains are presented. Furthermore we discuss possible future applications to eigenvalue enclosing/excluding of Schrödinger operators, hopefully even in spectral gaps.

In numerical verification methods for solutions of nonlinear fourth order elliptic equations, it is important to find the crucial constants in constructive a priori and a posteriori error estimates for finite element approximations to bi-harmonic problems. We develop procedures to accomplish this problem by verified computational techniques using Hermite spline functions for two dimensional rectangular domains. Several numerical examples which confirm the actual effectiveness of the method are presented.

We present a method to enclose fundamental solutions of linear ordinary differential equations, especially for a one dimensional Schrödinger equation which has a periodic potential. Our method is based on Floquet theory and our verification method for nonlinear equations. We show how to enclose fundamental solutions together with characteristic exponents and give a numerical example.

We consider an eigenvalue problem for differential operators, and show how guaranteed bounds
for eigenvalues (together with eigenvectors) are obtained and how non-existence of eigenvalues in a concrete region can be assured. Some examples for several types of operators are presented.

We prove the stability of a functional equation for real valued functions defined on a square-symmetric groupoid with a left unit element, and of an equation for real valued functions defined on an Abelian group.

This paper describes a computer-assisted stability proof for the Orr-Sommerfeld problem with Poiseuille flow. It is an application of a numerical verification technique for second-order elliptic boundary value problems introduced by a part of the authors.

An enclosure method for complex eigenvalues is presented. We formulate the eigenvalue problem as a nonlinear system and apply a fixed point theorem to enclose eigenvalues and eigenfunctions or basis of the corresponding invariant subspaces in case of multiple eigenvalues. Some enclosure examples are given.

Subject of investigation in this paper is a 1D-Schrödinger equation, where the potential is a sum of a periodic function and a perturbation decaying at $\pm \infty$. It is well known that the essential spectrum consists of spectral bands, and that there may or may not be additional eigenvalues below the lowest band or in the gaps between the bands. While enclosures for gap eigenvalues can comparably easily be obtained from numerical approximation, e.g. by D. Weinstein’s bounds, there seems to be no method available so far which is able to exclude eigenvalues in spectral gaps, i.e. which identifies sub-regions (of a gap) which contain no eigenvalues. Here, we propose such a method. It makes heavy use of computer assistance; nevertheless, the results are completely rigorous in the strict mathematical sense, since all computational errors are taken into account.

We propose a numerical method to enclose a solution of FitzHugh-Nagumo equation with Neumann boundary conditions. We construct, on the computer, a set which satisfies the hypothesis
of Schauder’s fixed point theorem for a compact map in a certain Sobolev space, and which therefore contains a solution. Several verified results are presented.


We consider the problem of verifying the existence of $H^1$ ground states of the 1D nonlinear Schrödinger equation for an interface of two periodic structures:

$$-u'' + V(x)u - \lambda u = \Gamma(x)|u|^{p-1}u \quad \text{on } \mathbb{R}$$

with $V(x) = V_1(x), \Gamma(x) = \Gamma_1(x)$ for $x > 0$ and $V(x) = V_2(x), \Gamma(x) = \Gamma_2(x)$ for $x < 0$. Here $V_1, V_2, \Gamma_1, \Gamma_2$ are periodic, $\lambda < \min \sigma(-\frac{d^2}{dx^2} + V)$, and $p > 1$. The article [T. Dohnal, M. Plum and W. Reichel, “Surface Gap Soliton Ground States for the Nonlinear Schrödinger Equation,” Comm. Math. Phys. 308, 511-542 (2011)] provides in the 1D case an existence criterion in the form of an integral inequality involving the linear potentials $V_1, V_2$ and the Bloch waves of the operators $-\frac{d^2}{dx^2} + V_1, V_2 - \lambda$. We choose here the classes of piecewise constant and piecewise linear potentials $V_{1,2}$ and check this criterion for a set of parameter values. In the piecewise constant case the Bloch waves are calculated explicitly and in the piecewise linear case verified enclosures of the Bloch waves are computed numerically. The integrals in the criterion are evaluated via interval arithmetic so that rigorous existence statements are produced. Examples of interfaces supporting ground states are reported including such, for which ground state existence follows for all periodic $\Gamma_{1,2}$ with $\text{ess sup } \Gamma_{1,2} > 0$.


This paper presents eigenvalue excluding methods for self-adjoint or non-self-adjoint eigenvalue problems in Hilbert spaces, including problems with partial differential operators. Eigenvalue exclosure means the determination of subsets of the complex field which do not contain eigenvalues of the given problem. Several verified eigenvalue excluding results for ordinary and partial differential operators are reported on.


This paper presents a computer-assisted procedure to prove the invertibility of a linear operator which is the sum of an unbounded bijective and a bounded operator in a Hilbert space, and to compute a bound for the norm of its inverse. By using some projection and constructive a priori error estimates, the invertibility condition together with the norm computation is formulated as an inequality based upon a method originally developed by the authors for obtaining existence and enclosure results for nonlinear partial differential equations. Several examples which confirm the actual effectiveness of the procedure are reported.


A compactness proof of a nonlinear operator related to stream function-vorticity formulation
for the Navier-Stokes equations is presented. The compactness of the operator provides important information for fixed-point formulations, especially for computer-assisted proofs based on Schauder’s fixed-point theorem. Our idea for the compactness proof comes from books by Girault & Raviart and Ladyzhenskaia, and our principle would be also applied to convex polygonal regions.

Time-variant fractional models are used to describe many applications, e.g. lithium-ion batteries. For such models, neither a controllability criterion for state space equations nor the energy-optimal control function are available so far. To overcome this limitation, in this paper a reachability and controllability definition for time-variant fractional state space systems is formulated and the analytical solution of the energy-optimal control problem is derived. In this context, a fractional Gramian-Matrix appears. The fact that the fractional system state must be interpreted differently than the state of a regular system is central to this paper. The reason for the difference in the interpretation is that the fractional system state additionally depends on an initialization procedure.

Time-variant fractional systems have many applications. For example, they can be used for system identification of lithium-ion batteries. However, the analytical solution of the time-variant fractional pseudo state space equation is missing so far. To overcome this limitation, this letter introduces a novel matrix approach, namely the generalized PeanoBaker series, which is comparable to the transition matrix in the case of ordinary systems. Using this matrix, the solution of the time-variant fractional pseudo state space equation is derived. The initialization process is taken into account, which has been proven to be a crucial part for fractional operator calculus. Following this initialization, a modified definition of a fractional pseudo state is presented.

We consider the fourth-order wave equation $\varphi_{tt} + \varphi_{xxxx} + f(\varphi) = 0$, $(x,t) \in \mathbb{R} \times \mathbb{R}$ with a nonlinearity $f$ vanishing at 0. Traveling waves $\varphi(x,t) = u(x-ct)$ satisfy the ODE $u'''' + c^2 u'' + f(u) = 0$ on $\mathbb{R}$, and for the case $f(u) = e^u - 1$, the existence of at least 36 solitary travelling waves was proved in [Breuer et al, Journal of Differential Equations (2016)] by computer assisted means. We investigate the orbital stability of these solutions via computation of their Morse indices and using results from [Grillakis et al, Journal of Functional Analysis (1987 & 1990)]. In order to achieve our results we make use of both analytical and computer-assisted techniques.

Proceedings

(1) N. Yamamoto, K. Nagatou, M. T. Nakao, Numerical verification method for elliptic eigenvalue


