In this talk, we discuss non-local operators like elliptic integrodifferential operators of fractional type
\[ Au := \text{p.v.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2\alpha}} \, dy \] (1)
or the Stokes operator with bounded measurable coefficients \( \mu \), formally given by
\[ Au := -\text{div}(\mu \nabla u) + \nabla \phi, \quad \text{div}(u) = 0 \text{ in } \mathbb{R}^d. \] (2)
These operators satisfy \( L^2 \)-resolvent estimates of the form
\[ \| (\lambda + A)^{-1} f \|_{L^2} \leq C \| f \|_{L^2} \quad (f \in L^2(\mathbb{R}^d)) \]
for \( \lambda \) in some complex sector \( \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta \} \). We describe how analogues of such a resolvent estimate can be established in \( L^p \) by virtue of certain non-local Caccioppoli inequalities. Such estimates build the foundation for many important functional analytic properties of these operators like maximal \( L^q \)-regularity.
More precisely, we establish resolvent estimates in \( L^p \) for \( p \) satisfying
\[ \left| \frac{1}{p} - \frac{1}{2} \right| < \frac{\alpha}{d} \]
in the case (1) and
\[ \left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d} \] (3)
in the case (2). This resembles a well-known situation for elliptic systems in divergence form with \( L^\infty \)-coefficients. Here, important estimates like Gaussian upper bounds for the semigroup cease to exist and the \( L^p \)-extrapolation has be concluded by other means.
In particular, for elliptic systems one can establish resolvent bounds for numbers \( p \) that satisfy (3) and if \( d \geq 3 \), Davies constructed examples which show that corresponding resolvent bounds do not hold for numbers \( 1 < p < \infty \) that satisfy
\[ \left| \frac{1}{p} - \frac{1}{2} \right| > \frac{1}{d} \]
These elliptic results give an indication that the result for the Stokes operator with \( L^\infty \)-coefficients is optimal as well.