Embedded eigenvalues for asymptotically periodic ODE systems

Seminar - KIT
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1 Problem setup and statement of main theorem

Consider the operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + A, \tag{1}$$

on the Hilbert space $L^2(\mathbb{R}, \mathbb{R}^n)$, where $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is a piecewise continuous matrix-valued function. Additionally, the matrix $A(x)$ is assumed to be asymptotically periodic, meaning that there exists a matrix-valued function $A_p(x)$ and a corresponding $p > 0$ such that $A_p(x+p) = A_p(x)$ for $x \in \mathbb{R}$, and $|A(x) - A_p(x)| \to 0$ as $|x| \to \infty$

We define the Banach space

$$X_\beta = \{B \in C(\mathbb{R}, \mathbb{R}^{n\times n}); B^T = B \text{ and } \|B\|_{X_\beta} = \sup_{x\in\mathbb{R}} |B(x)|(1+|x|)^\beta < \infty\}, \quad \beta > 1,$$

and consider instead the perturbed operator $\mathcal{L} + B$, defined by

$$(\mathcal{L} + B)\mathbf{u} = -\frac{d^2\mathbf{u}}{dx^2} + (A(x) + B(x))\mathbf{u}.$$

It can be written as a system of first order ODEs, taking $\mathbf{u} = \mathbf{u}_1$ and $\mathbf{u}' = \mathbf{u}_2$, obtaining

$$U' = \mathcal{A}(x; \lambda, B)U,$$

where $U = (\mathbf{u}_1, \mathbf{u}_2)^T \in \mathbb{R}^{2n}$ and

$$\mathcal{A}(x; \lambda, B) = \begin{bmatrix} 0 & I \\ A(x) + B(x) - \lambda I & 0 \end{bmatrix}.$$

We assume that

i) λ_0 is a simple embedded eigenvalue of $\mathcal{L}$

ii) $\|A - A_p\|_{X_\beta} < \infty$, for some $\beta > 1$.

iii) $e^{p\lambda_0} \notin \sigma(\Phi(p))$, where $\Phi(x)$ is the fundamental matrix solution to a first order system of ODEs with coefficient matrix

$$\mathcal{A}_p(x; \lambda_0) = \mathcal{A}(x; \lambda_0, 0) = \begin{bmatrix} 0 & I \\ A_p(x) - \lambda_0 I & 0 \end{bmatrix},$$

We denote by $2m$ the number of eigenvalues, including multiplicity, of $\Phi(p)$, with $\Phi(x)$ as iii), situated on the unit circle.

Theorem 1 (Main theorem)**.** *Let λ_0 be an eigenvalue of the unperturbed operator $\mathcal{L}$. Assume assumptions i)-iii) holds. Further, let*

$$\mathscr{S}_\varepsilon = \{B \in X_\beta | \exists \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon); \lambda \text{ eigenvalue of } \mathcal{L} + B\}.$$

Then there exists an $\varepsilon > 0$ and a neighbourhood $\mathcal{U}$ of $0 \in X_\beta$, such that $\mathscr{S}_\varepsilon \cap \mathcal{U}$ is a manifold of codimension $2m$ in X_β.

2 Preliminaries

Since $A(x) - A_p(x) \to 0$ and $B(x) \to 0$ as $|x| \to \infty$, by replacing the coefficient matrix by the periodic background matrix, we obtain the system at infinity

$$U' = \mathcal{A}_p(x; \lambda)U.$$

We can now decompose our coefficient matrix into two as

$$U' = (\mathcal{A}_p(x; \lambda_0) + L(x; \lambda, B))U,$$

where

$$L(x; \lambda, B) = \begin{bmatrix} 0 & 0 \\ A(x) - A_p(x) + B(x) + (\lambda_0 - \lambda)I & 0 \end{bmatrix}.$$

Theorem 2 (Floquet's theorem)**.** *Let $C(x)$ be a (piecewise) continuous periodic matrix function with period p, and $\Phi(x)$ be the fundamental matrix to the system*

$$\mathbf{y}' = C(x)\mathbf{y}.$$

Then there exists a non-singular piecewise differentiable matrix function $G(x)$ with period p, and a constant, possibly complex, matrix R such that

$$\Phi(x) = G(x)e^{Rx}, \text{ for all } x \in \mathbb{R}.$$

We can then use Floquet's theorem by setting

$$V(x) = G(x)^{-1}U(x)$$

which transforms the system into an asymptotically autonomous one

$$V' = (R(\lambda_0) + S(x; \lambda, B))V, \tag{2}$$

where $S(x; \lambda, B) = G(x)^{-1}L(x; \lambda, B)G(x)$.
Clearly, $S(x; \lambda_0, B) \to 0$ as $|x| \to \infty$. The transformed system at infinity can be expressed as

$$V' = R(\lambda)V.$$

Definition 1 (Exponential dichotomies)**.** *Let J be an unbounded interval on $\mathbb{R}$. An ODE system $U' = C(x)U$ is said to possess an exponential dichotomy on J if there exist constants $K > 0$, $\kappa^s < 0 < \kappa^u$ and a family of projections $P(x_0)$ such that:*

- *For any $x \in \mathbb{R}$ and $U \in \mathbb{R}^N$, there exists a unique (mild) solution $\Phi^s(x, x_0)U$ of the system defined for $x \geq x_0$, $x, x_0 \in J$ such that*

$$\Phi^s(x_0, x_0)U = P(x_0)U \text{ and } \|\Phi^s(x, x_0)U\| \leq Ke^{\kappa^s(x-x_0)}\|U\|.$$

- *For any $x \in \mathbb{R}$ and $U \in \mathbb{R}^N$, there exists a unique (mild) solution $\Phi^u(x, x_0)U$ of the system defined for $x \le x_0$, $x, x_0 \in J$ such that*

$$\Phi^u(x_0, x_0)U = \big(I - P(x_0)\big)U \text{ and } \|\Phi^u(x, x_0)U\| \le Ke^{\kappa^u(x-x_0)}\|U\|.$$

- *The solutions $\Phi^s(x, x_0)U$ and $\Phi^u(x, x_0)U$ satisfy*

$$\Phi^s(x, x_0)U \in RanP(x) \text{ for all } x \ge x_0, \quad x, x_0 \in J$$
$$\Phi^u(x, x_0)U \in kerP(x) \text{ for all } x \le x_0, \quad x, x_0 \in J.$$

Theorem 3 (Roughness theorem). *(i) If $U' = C(x)U$ possesses an exponential dichotomy on $\mathbb{R}_+$ with rates $\kappa^s < 0 < \kappa^u$ and constant $K > 0$ as in the definition, and if for some $T > 0$, $|D(x)| < \delta$ for all $x \ge T$, where $\delta \in (0, \min(-\kappa^s, \kappa^u)/2K)$, then the perturbed system $U' = (C(x) + D(x))U$ also possesses an exponential dichotomy on $\mathbb{R}_+$ with rates $\tilde{\kappa}^s = \kappa^s + 2K\delta < 0$, $\tilde{\kappa}^u = \kappa^u - 2K\delta > 0$ and some constant $\tilde{K} > 0$.*

(ii) If $U' = C(x)U$ possesses an exponential dichotomy on $\mathbb{R}_-$ with rates $\kappa^s < 0 < \kappa^u$ and constant $K > 0$ as in the definition, and if for some $T > 0$, $|D(x)| < \delta$ for all $x \le -T$, where $\delta \in (0, \min(-\kappa^s, \kappa^u)/2K)$, then the perturbed system $U' = (C(x) + D(x))U$ also possesses an exponential dichotomy on $\mathbb{R}_-$ with rates $\tilde{\kappa}^s = \kappa^s + 2K\delta < 0$, $\tilde{\kappa}^u = \kappa^u - 2K\delta > 0$ and some constant $\tilde{K} > 0$.

Lemma 1. *Let λ be an eigenvalue of the perturbed operator $\mathcal{L}_B$, and assume that assumptions i)-iii) hold. Further, let $\mathbf{u} \in L^2(\mathbb{R}; \mathbb{R}^n)$ be the corresponding eigenfunction. Denote by V the solution of (2) corresponding to $\mathbf{u}$. Then, there exists a positive constant K and $\hat{\kappa}$ such that*

$$|V(x)| \le Ke^{-\hat{\kappa}|x|} \text{ for all } x \in \mathbb{R}.$$

3 Main result

We define the stable and unstable subspaces E_+^s and E_-^u respectively. They consist of initial conditions for which the unperturbed system decays exponentially for increasing and decreasing values of x and are defined as

$$E_+^s = \{V \in \mathbb{R}^{2n};\ \ P^s(T; \lambda_0, 0)V = V\},$$
$$E_-^u = \{V \in \mathbb{R}^{2n};\ \ P^u(-T; \lambda_0, 0)V = V\}.$$

We further define the mapping $\iota : E_+^s \times E_-^u \times \mathbb{R} \times X_\beta \to \mathbb{R}^{2n}$ by

$$\iota(V_0^s, V_0^u; \lambda, B) = \Phi(0, T; \lambda, B)P^s(T; \lambda, B)V_0^s - \Phi(0, -T; \lambda, B)P^u(-T; \lambda, B)V_0^u.$$

We show that λ is an eigenvalue of the perturbed operator if and only if there exists a non-trivial pair $(V_0^s, V_0^u) \in E_+^s \times E_-^u$ such that

$$\iota(V_0^s, V_0^u; \lambda, B) = 0.$$

Let Q be a projection in $\mathbb{R}^{2n}$ onto $\text{Ran}\iota(\cdot, \cdot; \lambda_0, 0) = \Phi(0, T; \lambda_0, 0)E_+^s + \Phi(0, -T; \lambda_0, 0)E_-^u$. Then

$$Q\iota(V_0^s, V_0^u, \lambda, B) = 0,$$
$$(I-Q)\iota(V_0^s, V_0^u, \lambda, B) = 0.$$

We show that

- $(\Phi(0, T; \lambda_0, 0)E_+^s + \Phi(0, -T; \lambda_0, 0)E_-^u) = 2m + 1$

- for (λ, B) close to $(\lambda_0, 0)$, $Q\iota(V_0^s, V_0^u, \lambda, B) = 0$ has a unique solution

$$(V_0^s, V_0^u) = (V_0^s(\lambda, B), V_0^u(\lambda, B))$$

- $(I - Q)\iota(V_0^s(\lambda, B), V_0^u(\lambda, B); \lambda, B) = 0$, gives us a smooth function $\lambda(B)$ in a neighbourhood of $B = 0$ such that $\lambda(0) = \lambda_0$.

Here, $Z_* = (-\mathbf{u}_*', \mathbf{u}_*)^T$, where $\mathbf{u}_*$ is the eigenfunction of $\mathcal{L}_B$, and $F(\lambda, B) = \iota(V_0^s(\lambda, B), V_0^u(\lambda, B); \lambda, B) = (I - Q)\iota(V_0^s(\lambda, B), V_0^u(\lambda, B); \lambda, B)$.

Outline of the proof

It is shown that $\dim(\ker Q^*) = \dim(\ker Q) = 2m + 1$, and $Z_*(0) \in \ker Q^*$, with $Z_*(0) = (-\mathbf{u}_*'(0), \mathbf{u}_*(0))^T$, and $\mathbf{u}_*$ is the eigenfunction of $\mathcal{L} + B$.

We set $W_0(0) = Z_*(0)$, and further define $W_k(0) \in \mathbb{R}^{2n}$ for $k = 1, ..., 2m$, with, such that $\{W_k(0); k = 0, ..., 2m\}$ is a basis for $\ker Q^*$.

Let W_k be the solution of the adjoint transformed unperturbed system, i.e., $W' = -(R(\lambda_0) + S(x; \lambda_0, 0))^* W$, with initial value $W_k(0)$.

Further, define $F_k : X_\beta \to \mathbb{R}$ by

$$F_k(B) = \langle W_k(0), F(\lambda(B), B)\rangle, \quad k = 1, ..., 2m,$$

with F as before.

If $F_k(B) = 0$ for some $B \in X_\beta$ for all $k = 1, ..., 2m$, then $F(\lambda(B), B) = 0$ since $\{W_k(0); k = 0, ..., 2m\}$ is a basis for $\ker Q^*$. The converse clearly holds as well.

Additionally, it can be shown that $F_k'(0)$ are all linearly independent, which essentially concludes the proof.

(For the curious)

Consider the decomposition $X_\beta = \ker \overline{F}'(0) \oplus X$, where $\overline{F}(B) = \big(F_1(B), ..., F_{2m}(B)\big)^T$ and X has dimension $2m$.

Then for all $B \in X_\beta$, we have $B = B_1 + B_2$, for $B_1 \in \ker \overline{F}'(0)$ and $B_2 \in X$.

Define the function $f : \ker \overline{F}'(0) \times X \to \mathbb{R}^{2m}$ by

$$f(B_1, B_2) = \overline{F}(B_1 + B_2) = \overline{F}(B).$$

Differentiating gives $\partial_{B_i} f(B_1, B_2)B_i = \overline{F}'(B_1 + B_2)B_i$ which implies that $\ker \partial_{B_1} f(0, 0) = \ker \overline{F}'(0)$ and $\ker \partial_{B_2} f(0, 0) = \{0\}$. By the implicit function theorem, $f(B_1, B_2) = 0$ defines B_2 as a smooth function, $g : U \subset \ker \overline{F}'(0) \to X$, of B_1 in a neighbourhood of $B_1 = 0, B_2 = 0$, where U is a neighbourhood of $0 \in \ker \overline{F}'(0)$.

Then $g(B_1) = B_2$ if and only if $f(B_1, B_2) = 0$, or equivalently, if and only if $\overline{F}(B) = 0$.

Further, let $\zeta : U \subset \ker \overline{F}'(0) \to \ker \overline{F}'(0) \times X$ be defined by $B_1 \mapsto (B_1, f(B_1))$.

Defining $G(x, y) = y - \zeta(x)$, we can apply the implicit function theorem again, giving the function $h(x) = y$, defined locally. This must then be ζ^{-1}, and we are done.