Embedded Eigenvalues for Asymptotically Periodic ODE Systems

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Background

- Persistance of eigenvalues, differences between isolated and embedded
- Follows the ideas of *Perturbations of embedded eigenvalues for self-adjoint ODE systems* by A. Papalazarou and S. Maad Sasane.

Problem setup

Consider the operator $\mathcal{L} : L^2(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ defined by

$$\mathcal{L}\mathbf{u} = -\frac{d^2\mathbf{u}}{dx^2} + A(x)\mathbf{u},$$

where $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is piecewiese continuous and diagonal. We further assume that A is asymptotically periodic. Meaning that there exists a periodic matrix valued function $A_p : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $p \in \mathbb{R}_+$ such that $A_p(x+p) = A_p(x)$ and $A(x) - A_p(x) \to 0$ as $|x| \to \infty$.
Additionally, we assume that $\mathcal{L}$ has a (simple) embedded eigenvalue, denoted by λ_0.

Problem setup

We then add a perturbation $B \in X_\beta$, $\beta > 1$, where

$$X_\beta = \{B \in C(\mathbb{R}, \mathbb{R}^{n\times n}); B^T = B \text{ and } \|B\|_{X_\beta} = \sup_{x\in\mathbb{R}} |B(x)|(1+|x|)^\beta < \infty\},$$

a Banach space.
We then consider instead the perturbed operator $\mathcal{L} + B$, defined by

$$(\mathcal{L} + B)\mathbf{u} = -\frac{d^2\mathbf{u}}{dx^2} + (A(x) + B(x))\mathbf{u}.$$

Main result

It can be written as a system of first order ODEs, taking $\mathbf{u} = \mathbf{u}_1$ and $\mathbf{u}' = \mathbf{u}_2$, obtaining

$$U' = \mathcal{A}(x; \lambda, B)U,$$

where $U = (\mathbf{u}_1, \mathbf{u}_2)^T \in \mathbb{R}^{2n}$ and

$$\mathcal{A}(x; \lambda, B) = \begin{bmatrix} 0 & I \\ A(x) + B(x) - \lambda I & 0 \end{bmatrix}.$$

Since $\mathcal{A}$ is piecewise continuous, we will consider so called mild solutions of the system, which are solutions, U, to the integral equation

$$U(x) = U(x_0) + \int_{x_0}^{x} \mathcal{A}(\xi; \lambda, B)U(\xi)d\xi.$$

Main result

We prove that, for $\varepsilon > 0$, the set

$$\mathscr{S}_\varepsilon = \{B \in X_\beta | \exists \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon); \lambda \text{ eigenvalue of } \mathcal{L} + B\}$$

form a manifold with a specified codimension in a neighbourhood of $0 \in X_\beta$.

Main result

To do this, we must also assume that

(i) $\|A - A_p\|_{X_\beta} < \infty$, for some $\beta > 1$

(ii) $e^{p\lambda_0} \notin \sigma(\Phi(p))$, where $\Phi(x)$ is the fundamental matrix solution to the system

$$U' = \begin{bmatrix} 0 & I \\ A_p(x) - \lambda_0 I & 0 \end{bmatrix} U, \quad U \in \mathbb{R}^{2n}$$

.

Further, we denote by $2m$ the number of eigenvalues, including multiplicity, of $\Phi(p)$, with $\Phi(x)$ as (ii), situated on the unit circle.

Main result

Theorem

Let λ_0 be an embedded eigenvalue to the unperturbed operator $\mathcal{L}$. Assume that the assumptions holds. Then there exists an $\varepsilon > 0$ and a neighbourhood $\mathcal{U}$ of $0 \in X_\beta$, such that $\mathscr{S}_\varepsilon \cap \mathcal{U}$ form a manifold of codimension $2m$ in X_β.

General structure

1. Introduction to Floquet theory
2. Application of Floquet theory
3. Introduction to exponential dichotomies
4. Application of exponential dichotomies
5. Proof of main theorem
6. Example

1. Introduction to Floquet theory

- Floquet theory is a branch of ODE theory dealing with linear first order systems with a piecewise continuous and periodic coefficient matrix.
- The spectrum for $\mathcal{L}$ in the case when $n = 1$ and $A = A_p$.

1. Introduction to Floquet theory

Theorem

The spectrum, $\sigma(\mathcal{L})$, in the case where $A = A_p$ and $n = 1$, can be written as a disjoint union of closed intervals

$$\sigma(\mathcal{L}) = \bigcup_{n \in I} \mathcal{I}_n,$$

where I is a finite or countably infinite index set and $\mathcal{I}_n$ are intervals of all λ corresponding to polynomially bounded solutions, called spectral bands. Further, the spectrum $\sigma(\mathcal{L})$ is purely continuous.

1. Introduction to Floquet theory

- Floquet theory is a branch of ODE theory dealing with linear first order systems with a piecewise continuous and periodic coefficient matrix.
- The spectrum for $\mathcal{L}$ in the case when $n = 1$ and $A = A_p$.
- An example on how we can construct eigenfunctions with $A \neq A_p$ in the $n = 1$ case.

For the first example, we shall use the coefficient function

$$A(x) = \begin{cases} -V_0, & k + a \le x \le k+1 \\ 0, & \text{otherwise} \end{cases}, \quad \text{for all } k \in \mathbb{Z},$$

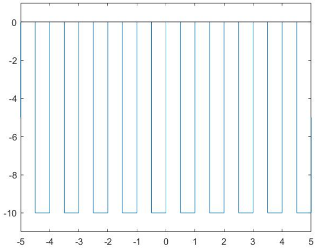

Figure: Coefficient function.

1. Introduction to Floquet theory

By fixing λ in one of the spectral gaps, and amplifying the amplitude of the coefficient function in the first period, we get the eigenfunction

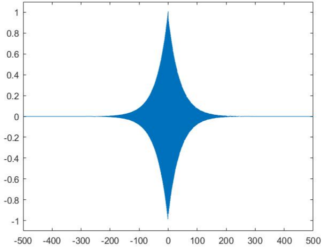

Figure: Eigenfunction example.

1. Introduction to Floquet theory

Another, continous, example, by a similar procedure, we can alter a (shifted) cosine wave by a Gaussian, giving the asymptotically periodic potential

$$V(x) = V_0(\cos(x) - 1) - V_p e^{-(x-s)^2}$$

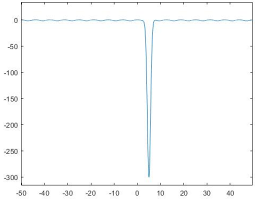

Figure: Continuous coefficient function.

Resulting in, as is somewhat expected, a very similar looking eigenfunction

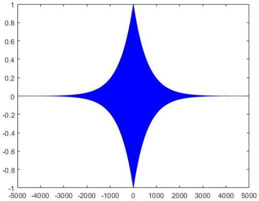

Figure: Eigenfunction example

2. Application of Floquet theory

Since $A(x) - A_p(x) \to 0$ and $B(x) \to 0$ as $|x| \to \infty$, by replacing the coefficient matrix by the periodic background matrix, we obtain the system at infinity

$$U' = \mathcal{A}_p(x;\lambda)U,$$

with

$$\mathcal{A}_p(x;\lambda) = \begin{bmatrix} 0 & I \\ A_p(x) - \lambda I & 0 \end{bmatrix}.$$

We can now decompose our coefficient matrix into two as

$$U' = (\mathcal{A}_p(x;\lambda_0) + L(x;\lambda,B))U,$$

where

$$L(x;\lambda,B) = \begin{bmatrix} 0 & 0 \\ A(x) - A_p(x) + B(x) + (\lambda_0 - \lambda)I & 0 \end{bmatrix}.$$

2. Application of Floquet theory

Theorem (Floquet's theorem)

Let $C(x)$ be a (piecewise) continuous periodic matrix function with period p, and $\Phi(x)$ be the fundamental matrix to the system

$$\mathbf{y}' = C(x)\mathbf{y}.$$

Then there exists a non-singular piecewise differentiable matrix function $G(x)$ with period p, and a constant, possibly complex, matrix R such that

$$\Phi(x) = G(x)e^{Rx}, \text{ for all } x \in \mathbb{R}.$$

2. Application of Floquet theory

We can then use Floquet's theorem by setting

$$V(x) = G(x)^{-1}U(x),$$

which transforms the system into an asymptotically autonomous one

$$V' = (R(\lambda_0) + S(x; \lambda, B))V,$$

where $S(x; \lambda, B) = G(x)^{-1}L(x; \lambda, B)G(x)$. Clearly, $S(x; \lambda_0, B) \to 0$ as $|x| \to \infty$. The transformed system at infinity can be expressed as

$$V' = R(\lambda)V.$$

3. Introduction to exponential dichotomies

Exponential dichotomies is a tool, originally introduced by Oskar Perron, used to investigate the stability properties and asymptotic behaviour of non-autonomous differential equations.

3. Introduction to exponential dichotomies

Definition

Let J be an unbounded interval on $\mathbb{R}$. An ODE system $U' = C(x)U$ is said to possess an exponential dichotomy on J if there exist constants $K > 0$, $\kappa^s < 0 < \kappa^u$ and a family of projections $P(x_0)$ such that:

- For any $x \in \mathbb{R}$ and $U \in \mathbb{R}^N$, there exists a unique (mild) solution $\Phi^s(x, x_0)U$ of the system defined for $x \geq x_0$, $x, x_0 \in J$ such that

$$\Phi^s(x_0, x_0)U = P(x_0)U \text{ and } \|\Phi^s(x, x_0)U\| \leq Ke^{\kappa^s(x-x_0)}\|U\|.$$

- For any $x \in \mathbb{R}$ and $U \in \mathbb{R}^N$, there exists a unique (mild) solution $\Phi^u(x, x_0)U$ of the system defined for $x \leq x_0$, $x, x_0 \in J$ such that

$$\Phi^u(x_0, x_0)U = \big(I - P(x_0)\big)U \text{ and } \|\Phi^u(x, x_0)U\| \leq Ke^{\kappa^u(x-x_0)}\|U\|.$$

3. Introduction to exponential dichotomies

Definition

- The solutions $\Phi^s(x,x_0)U$ and $\Phi^u(x,x_0)U$ satisfy

$$\Phi^s(x,x_0)U \in \mathrm{Ran} P(x) \text{ for all } x \geq x_0, \quad x, x_0 \in J$$
$$\Phi^u(x,x_0)U \in \ker P(x) \text{ for all } x \leq x_0, \quad x, x_0 \in J.$$

3. Introduction to exponential dichotomies

We can use a roughness theorem for the exponential dichotomies to extend it to the perturbed case.

3. Introduction to exponential dichotomies

Theorem (Roughness theorem)

(i) *If* $U' = C(x)U$ *possesses an exponential dichotomy on* $\mathbb{R}_+$ *with rates* $\kappa^s < 0 < \kappa^u$ *and constant* $K > 0$ *as in the definition, and if for some* $T > 0$, $|D(x)| < \delta$ *for all* $x \geq T$, *where* $\delta \in (0, \min(-\kappa^s, \kappa^u)/2K)$, *then the perturbed system* $U' = (C(x) + D(x))U$ *also possesses an exponential dichotomy on* $\mathbb{R}_+$ *with rates* $\tilde{\kappa}^s = \kappa^s + 2K\delta < 0$, $\tilde{\kappa}^u = \kappa^u - 2K\delta > 0$ *and some constant* $\tilde{K} > 0$.

(ii) *If* $U' = C(x)U$ *possesses an exponential dichotomy on* $\mathbb{R}_-$ *with rates* $\kappa^s < 0 < \kappa^u$ *and constant* $K > 0$ *as in the definition, and if for some* $T > 0$, $|D(x)| < \delta$ *for all* $x \leq -T$, *where* $\delta \in (0, \min(-\kappa^s, \kappa^u)/2K)$, *then the perturbed system* $U' = (C(x) + D(x))U$ *also possesses an exponential dichotomy on* $\mathbb{R}_-$ *with rates* $\tilde{\kappa}^s = \kappa^s + 2K\delta < 0$, $\tilde{\kappa}^u = \kappa^u - 2K\delta > 0$ *and some constant* $\tilde{K} > 0$.

5. Applications of exponential dichotomies

- We show that our transformed system at infinity possess an exponential dichotomy.
- Show that the system with the added perturbation also possess an exponential dichotomy.
- Show that the eigenfunctions of our perturbed operator decay exponentially.

6. Main result

- Lyapunov-Schmidt reduction.
- Proof of main theorem.

6. Main result

We define the stable and unstable subspaces E_+^s and E_-^u respectively. They consist of initial conditions for which the unperturbed system decays exponentially for increasing and decreasing values of x and are defined as

$$E_+^s = \{V \in \mathbb{R}^{2n};\ \ P^s(T;\lambda_0,0)V = V\},$$
$$E_-^u = \{V \in \mathbb{R}^{2n};\ \ P^u(-T;\lambda_0,0)V = V\}.$$

We further define the mapping $\iota : E_+^s \times E_-^u \times \mathbb{R} \times X_\beta \to \mathbb{R}^{2n}$ by

$$\begin{aligned}\iota(V_0^s, V_0^u; \lambda, B) =& \Phi(0,T;\lambda,B)P^s(T;\lambda,B)V_0^s \\ &- \Phi(0,-T;\lambda,B)P^u(-T;\lambda,B)V_0^u.\end{aligned}$$

6. Main result

We show that λ is an eigenvalue of the perturbed operator if and only if there exists a non-trivial pair $(V_0^s, V_0^u) \in E_+^s \times E_-^u$ such that

$$\iota(V_0^s, V_0^u; \lambda, B) = 0.$$

Let Q be a projection in $\mathbb{R}^{2n}$ onto
Ran$\iota(\cdot, \cdot; \lambda_0, 0) = \Phi(0, T; \lambda_0, 0)E_+^s + \Phi(0, -T; \lambda_0, 0)E_-^u$. Then

$$Q\iota(V_0^s, V_0^u, \lambda, B) = 0,$$
$$(I - Q)\iota(V_0^s, V_0^u, \lambda, B) = 0.$$

6. Main result

We show that

- $\operatorname{codim}(\Phi(0,T;\lambda_0,0)E_+^s + \Phi(0,-T;\lambda_0,0)E_-^u) = 2m+1$
- for (λ, B) close to $(\lambda_0, 0)$, $Q\iota(V_0^s, V_0^u, \lambda, B) = 0$ has a unique solution

$$(V_0^s, V_0^u) = (V_0^s(\lambda, B), V_0^u(\lambda, B))$$

- $(I-Q)\iota(V_0^s(\lambda, B), V_0^u(\lambda, B); \lambda, B) = 0$, gives us a smooth function $\lambda(B)$ in a neighbourhood of $B = 0$ such that $\lambda(0) = \lambda_0$.

6. Main result

By means of the implicit function theorem, through Lyapunov Schmidt reduction, we now have V_0^s, V_0^u and λ as smooth functions of B in a neighbourhood around $0 \in X_\beta$.

We can now prove our main theorem.

6. Main results

It is shown that $\dim(\ker Q^*) = \dim(\ker Q) = 2m + 1$, and $Z_*(0) \in \ker Q^*$, with $Z_*(0) = (-\mathbf{u}'_*(0), \mathbf{u}_*(0))^T$, and $\mathbf{u}_*$ is the eigenfunction of $\mathcal{L} + B$.

We set $W_0(0) = Z_*(0)$, and further define $W_k(0) \in \mathbb{R}^{2n}$ for $k = 1, ..., 2m$, with, such that $\{W_k(0); k = 0, ..., 2m\}$ is a basis for $\ker Q^*$.

Let W_k be the solution of the adjoint transformed unperturbed system, i.e., $W' = -(R(\lambda_0) + S(x; \lambda_0, 0))^* W$, with initial value $W_k(0)$.

6. Main result

Further, define $F_k : X_\beta \to \mathbb{R}$ by

$$F_k(B) = \langle W_k(0), F(\lambda(B), B) \rangle, \quad k = 1, ..., 2m,$$

with $F(\lambda, B) = \iota(V_0^s(\lambda(B), B), V_0^u(\lambda(B), B); \lambda(B), B)$.

If $F_k(B) = 0$ for some $B \in X_\beta$ for all $k = 1, ..., 2m$, then $F(\lambda(B), B) = 0$ since $\{W_k(0); k = 0, ..., 2m\}$ is a basis for $\ker Q^*$. The converse clearly holds as well.
Additionally, it can be shown that $F_k'(0)$ are all linearly independent

6. Main result

Consider the decomposition $X_\beta = \ker \overline{F}'(0) \oplus X$, where $\overline{F}(B) = \big(F_1(B), ..., F_{2m}(B)\big)^T$ and X has dimension $2m$.

Then for all $B \in X_\beta$, we have $B = B_1 + B_2$, for $B_1 \in \ker \overline{F}'(0)$ and $B_2 \in X$.

Define the function $f : \ker \overline{F}'(0) \times X \to \mathbb{R}^{2m}$ by

$$f(B_1, B_2) = \overline{F}(B_1 + B_2) = \overline{F}(B).$$

Differentiating gives $\partial_{B_i} f(B_1, B_2)B_i = \overline{F}'(B_1 + B_2)B_i$ which implies that $\ker \partial_{B_1} f(0,0) = \ker \overline{F}'(0)$ and $\ker \partial_{B_2} f(0,0) = \{0\}$.

6. Main result

By the implicit function theorem, $f(B_1, B_2) = 0$ defines B_2 as a smooth function, $g : U \subset \ker \overline{F}'(0) \to X$, of B_1 in a neighbourhood of $B_1 = 0, B_2 = 0$, where U is a neighbourhood of $0 \in \ker \overline{F}'(0)$.

Then $g(B_1) = B_2$ if and only if $f(B_1, B_2) = 0$, or equivalently, if and only if $\overline{F}(B) = 0$.

Further, let $\zeta : U \subset \ker \overline{F}'(0) \to \ker \overline{F}'(0) \times X$ be defined by $B_1 \mapsto (B_1, f(B_1))$.

Defining $G(x, y) = y - \zeta(x)$, we can apply the implicit function theorem again, giving the function $h(x) = y$, defined locally. This must then be ζ^{-1}, and we are done.

7. Example

We choose

$$A(x) = \begin{bmatrix} V_1(x) & 0 \\ 0 & V_2(x) \end{bmatrix},$$

with $V_1(x)$ as in the last example and $V_2(x) = 0$. We choose λ_0 as in the last example, so that for the first equation, there is a corresponding eigenfunction, u_1, as before. The spectrum for the second equation is purely continuous and covers, in particular, λ_0. That means that λ_0 is an embedded eigenvalue.

The corresponding eigenfunction for the system is $(u_1, 0)^T$.

7. Example

The function $F_k(B)$ can be equivalently written as

$$F_k(B) = -\int_{-\infty}^{\infty} \langle \mathbf{z}_k(\xi), (B - (\lambda(B) - \lambda_0)I)\mathbf{u}(\xi; B)\rangle d\xi,$$

with $\mathbf{z}_k$ as $(G^*)^{-1}W_k = Z_k = (-\mathbf{z}_k', \mathbf{z}_k)^T$. Giving

$$F_k'(0)B =$$

$$-\int_{-\infty}^{\infty} \langle \mathbf{z}_k, B(\xi)\mathbf{u}_*(\xi)\rangle d\xi - \int_{-\infty}^{\infty} \langle \mathbf{u}_*(\xi), B(\xi)\mathbf{u}_*(\xi)\rangle d\xi \int_{-\infty}^{\infty} \langle \mathbf{z}_k(\xi), \mathbf{u}_*(\xi)\rangle d\xi$$

7. Example

In this example, for $\mathbf{z}_1$ and $\mathbf{z}_2$, we choose some generalized eigenfunctions

$$\mathbf{z}_1(x) = \begin{bmatrix} 0 \\ z_1(x) \end{bmatrix} \quad \text{and} \quad \mathbf{z}_2(x) = \begin{bmatrix} 0 \\ z_2(x) \end{bmatrix}.$$

Since

$$B = \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{12}(x) & b_{22}(x) \end{bmatrix}$$

we have

$$\begin{aligned} \langle \mathbf{z}_k(\xi), B(\xi)\mathbf{u}_*(\xi) \rangle &= \begin{bmatrix} 0 & z_k(\xi) \end{bmatrix} \begin{bmatrix} b_{11}(\xi) & b_{12}(\xi) \\ b_{12}(\xi) & b_{22}(\xi) \end{bmatrix} \begin{bmatrix} u_1(\xi) \\ 0 \end{bmatrix} \\ &= b_{12}(\xi) z_k(\xi) u_1(\xi), \end{aligned}$$

7. Example

It follows that

$$F_k'(0)B = \int_{-\infty}^{\infty} b_{12}(\xi) z_k(\xi) u_1(\xi) d\xi \quad \text{for } k = 1, 2.$$

Hence, the manifold $\mathscr{M}$ is tangent to the subspace of perturbations $B \in X_\beta$ such that the off-diagonal elements are orthogonal to $z_k(x)u_k(x)$. This follows since $\mathscr{M}$ is described in a neighbourhood of $B = 0$ by the equations $F_k(B) = 0$, and that the eigenvalue can only persist if $B \in \mathscr{M}$.

Thank you all for listening!