Persistance of eigenvalues, differences between isolated and embedded

Follows the ideas of *Perturbations of embedded eigenvalues for self-adjoint ODE systems* by A. Papalazarou and S. Maad Sasane.
Consider the operator $\mathcal{L} : L^2(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ defined by

$$\mathcal{L}u = -\frac{d^2u}{dx^2} + A(x)u,$$

where $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is piecewisewise continuous and diagonal. We further assume that $A$ is asymptotically periodic. Meaning that there exists a periodic matrix valued function $A_p : \mathbb{R} \to \mathbb{R}^{n \times n}$ and $p \in \mathbb{R}_+$ such that $A_p(x + p) = A_p(x)$ and $A(x) - A_p(x) \to 0$ as $|x| \to \infty$.

Additionally, we assume that $\mathcal{L}$ has a (simple) embedded eigenvalue, denoted by $\lambda_0$. 
Problem setup

We then add a perturbation $B \in X_\beta$, $\beta > 1$, where

$$X_\beta = \{ B \in C(\mathbb{R}, \mathbb{R}^{n \times n}); B^T = B \text{ and } \| B \|_{X_\beta} = \sup_{x \in \mathbb{R}} |B(x)|(1+|x|)^\beta < \infty \},$$

a Banach space.

We then consider instead the perturbed operator $\mathcal{L} + B$, defined by

$$(\mathcal{L} + B)u = -\frac{d^2 u}{dx^2} + (A(x) + B(x))u.$$
Main result

It can be written as a system of first order ODEs, taking \( u = u_1 \) and \( u' = u_2 \), obtaining

\[
U' = A(x; \lambda, B)U,
\]

where \( U = (u_1, u_2)^T \in \mathbb{R}^{2n} \) and

\[
A(x; \lambda, B) = \begin{bmatrix}
0 & I \\
A(x) + B(x) - \lambda I & 0
\end{bmatrix}.
\]

Since \( A \) is piecewise continuous, we will consider so called mild solutions of the system, which are solutions, \( U \), to the integral equation

\[
U(x) = U(x_0) + \int_{x_0}^{x} A(\xi; \lambda, B)U(\xi) d\xi.
\]
Main result

We prove that, for $\varepsilon > 0$, the set

$$\mathcal{S}_\varepsilon = \{ B \in X_\beta | \exists \lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon); \lambda \text{ eigenvalue of } L + B \}$$

form a manifold with a specified codimension in a neighbourhood of $0 \in X_\beta$. 
To do this, we must also assume that

(i) \( \| A - A_P \|_{X_\beta} < \infty \), for some \( \beta > 1 \)

(ii) \( e^{p\lambda_0} \notin \sigma(\Phi(p)) \), where \( \Phi(x) \) is the fundamental matrix solution to the system

\[
U' = \begin{bmatrix} 0 & I \\ A_p(x) - \lambda_0 I & 0 \end{bmatrix} U, \quad U \in \mathbb{R}^{2n}
\]

Further, we denote by \( 2m \) the number of eigenvalues, including multiplicity, of \( \Phi(p) \), with \( \Phi(x) \) as (ii), situated on the unit circle.
Main result

**Theorem**

Let $\lambda_0$ be an embedded eigenvalue to the unperturbed operator $\mathcal{L}$. Assume that the assumptions holds. Then there exists an $\varepsilon > 0$ and a neighbourhood $\mathcal{U}$ of $0 \in X_\beta$, such that $\mathcal{S}_\varepsilon \cap \mathcal{U}$ form a manifold of codimension $2m$ in $X_\beta$. 
General structure

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2. Application of Floquet theory
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4. Application of exponential dichotomies
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6. Example
1. Introduction to Floquet theory

- Floquet theory is a branch of ODE theory dealing with linear first order systems with a piecewise continuous and periodic coefficient matrix.
- The spectrum for $L$ in the case when $n = 1$ and $A = A_p$. 

1. Introduction to Floquet theory

**Theorem**

The spectrum, $\sigma(\mathcal{L})$, in the case where $A = A_p$ and $n = 1$, can be written as a disjoint union of closed intervals

$$\sigma(\mathcal{L}) = \bigcup_{n \in I} \mathcal{I}_n,$$

where $I$ is a finite or countably infinite index set and $\mathcal{I}_n$ are intervals of all $\lambda$ corresponding to polynomially bounded solutions, called spectral bands. Further, the spectrum $\sigma(\mathcal{L})$ is purely continuous.
1. Introduction to Floquet theory

- Floquet theory is a branch of ODE theory dealing with linear first order systems with a piecewise continuous and periodic coefficient matrix.
- The spectrum for $\mathcal{L}$ in the case when $n = 1$ and $A = A_p$.
- An example on how we can construct eigenfunctions with $A \neq A_p$ in the $n = 1$ case.
1. Introduction to Floquet theory

For the first example, we shall use the coefficient function

\[ A(x) = \begin{cases} 
-V_0, & k + a \leq x \leq k + 1 \\
0, & \text{otherwise} 
\end{cases}, \quad \text{for all } k \in \mathbb{Z}, \]

Figure: Coefficient function.
1. Introduction to Floquet theory

By fixing $\lambda$ in one of the spectral gaps, and amplifying the amplitude of the coefficient function in the first period, we get the eigenfunction

**Figure:** Eigenfunction example.
Another, continuous, example, by a similar procedure, we can alter a (shifted) cosine wave by a Gaussian, giving the asymptotically periodic potential

\[ V(x) = V_0(\cos(x) - 1) - V_p e^{-(x-s)^2} \]

**Figure:** Continuous coefficient function.
1. Introduction to Floquet theory

Resulting in, as is somewhat expected, a very similar looking eigenfunction

Figure: Eigenfunction example
2. Application of Floquet theory

Since $A(x) - A_p(x) \to 0$ and $B(x) \to 0$ as $|x| \to \infty$, by replacing the coefficient matrix by the periodic background matrix, we obtain the system at infinity

$$U' = A_p(x; \lambda)U,$$

with

$$A_p(x; \lambda) = \begin{bmatrix} 0 & I \\ A_p(x) - \lambda I & 0 \end{bmatrix}.$$

We can now decompose our coefficient matrix into two as

$$U' = (A_p(x; \lambda_0) + L(x; \lambda, B))U,$$

where

$$L(x; \lambda, B) = \begin{bmatrix} 0 & 0 \\ A(x) - A_p(x) + B(x) + (\lambda_0 - \lambda)I & 0 \end{bmatrix}.$$
2. Application of Floquet theory

**Theorem (Floquet’s theorem)**

Let $C(x)$ be a (piecewise) continuous periodic matrix function with period $p$, and $\Phi(x)$ be the fundamental matrix to the system

$$y' = C(x)y.$$

Then there exists a non-singular piecewise differentiable matrix function $G(x)$ with period $p$, and a constant, possibly complex, matrix $R$ such that

$$\Phi(x) = G(x)e^{Rx}, \text{ for all } x \in \mathbb{R}.$$
We can then use Floquet’s theorem by setting

\[ V(x) = G(x)^{-1}U(x), \]

which transforms the system into an asymptotically autonomous one

\[ V' = (R(\lambda_0) + S(x; \lambda, B))V, \]

where \( S(x; \lambda, B) = G(x)^{-1}L(x; \lambda, B)G(x) \). Clearly, \( S(x; \lambda_0, B) \to 0 \) as \( |x| \to \infty \). The transformed system at infinity can be expressed as

\[ V' = R(\lambda)V. \]
3. Introduction to exponential dichotomies

Exponential dichotomies is a tool, originally introduced by Oskar Perron, used to investigate the stability properties and asymptotic behaviour of non-autonomous differential equations.
3. Introduction to exponential dichotomies

Definition

Let $J$ be an unbounded interval on $\mathbb{R}$. An ODE system $U' = C(x)U$ is said to possess an exponential dichotomy on $J$ if there exist constants $K > 0$, $\kappa^s < 0 < \kappa^u$ and a family of projections $P(x_0)$ such that:

- For any $x \in \mathbb{R}$ and $U \in \mathbb{R}^N$, there exists a unique (mild) solution $\Phi^s(x, x_0)U$ of the system defined for $x \geq x_0$, $x, x_0 \in J$ such that

  \[ \Phi^s(x_0, x_0)U = P(x_0)U \text{ and } \|\Phi^s(x, x_0)U\| \leq Ke^{\kappa^s(x-x_0)}\|U\|. \]

- For any $x \in \mathbb{R}$ and $U \in \mathbb{R}^N$, there exists a unique (mild) solution $\Phi^u(x, x_0)U$ of the system defined for $x \leq x_0$, $x, x_0 \in J$ such that

  \[ \Phi^u(x_0, x_0)U = (I - P(x_0))U \text{ and } \|\Phi^u(x, x_0)U\| \leq Ke^{\kappa^u(x-x_0)}\|U\|. \]
3. Introduction to exponential dichotomies

Definition

- The solutions \( \Phi^s(x, x_0)U \) and \( \Phi^u(x, x_0)U \) satisfy

\[
\begin{align*}
\Phi^s(x, x_0)U &\in \text{Ran} P(x) \text{ for all } x \geq x_0, \quad x, x_0 \in J \\
\Phi^u(x, x_0)U &\in \text{ker} P(x) \text{ for all } x \leq x_0, \quad x, x_0 \in J.
\end{align*}
\]
3. Introduction to exponential dichotomies

We can use a roughness theorem for the exponential dichotomies to extend it to the perturbed case.
3. Introduction to exponential dichotomies

**Theorem (Roughness theorem)**

(i) If $U' = C(x)U$ possesses an exponential dichotomy on $\mathbb{R}_+$ with rates $\kappa^s < 0 < \kappa^u$ and constant $K > 0$ as in the definition, and if for some $T > 0$, $|D(x)| < \delta$ for all $x \geq T$, where $\delta \in (0, \min(-\kappa^s, \kappa^u)/2K)$, then the perturbed system $U' = (C(x) + D(x))U$ also possesses an exponential dichotomy on $\mathbb{R}_+$ with rates $\tilde{\kappa}^s = \kappa^s + 2K\delta < 0$, $\tilde{\kappa}^u = \kappa^u - 2K\delta > 0$ and some constant $\tilde{K} > 0$.

(ii) If $U' = C(x)U$ possesses an exponential dichotomy on $\mathbb{R}_-$ with rates $\kappa^s < 0 < \kappa^u$ and constant $K > 0$ as in the definition, and if for some $T > 0$, $|D(x)| < \delta$ for all $x \leq -T$, where $\delta \in (0, \min(-\kappa^s, \kappa^u)/2K)$, then the perturbed system $U' = (C(x) + D(x))U$ also possesses an exponential dichotomy on $\mathbb{R}_-$ with rates $\tilde{\kappa}^s = \kappa^s + 2K\delta < 0$, $\tilde{\kappa}^u = \kappa^u - 2K\delta > 0$ and some constant $\tilde{K} > 0$. 
5. Applications of exponential dichotomies

- We show that our transformed system at infinity possess an exponential dichotomy.
- Show that the system with the added perturbation also possess an exponential dichotomy.
- Show that the eigenfunctions of our perturbed operator decay exponentially.
6. Main result

- Lyapunov-Schmidt reduction.
- Proof of main theorem.
6. Main result

We define the stable and unstable subspaces $E^s_+$ and $E^u_-$ respectively. They consist of initial conditions for which the unperturbed system decays exponentially for increasing and decreasing values of $x$ and are defined as

$$E^s_+ = \{ V \in \mathbb{R}^{2n}; \; P^s(T; \lambda_0, 0)V = V \},$$
$$E^u_- = \{ V \in \mathbb{R}^{2n}; \; P^u(-T; \lambda_0, 0)V = V \}.$$

We further define the mapping $\iota : E^s_+ \times E^u_- \times \mathbb{R} \times X_\beta \rightarrow \mathbb{R}^{2n}$ by

$$\iota(V^s_0, V^u_0; \lambda, B) = \Phi(0, T; \lambda, B)P^s(T; \lambda, B)V^s_0 - \Phi(0, -T; \lambda, B)P^u(-T; \lambda, B)V^u_0.$$
We show that $\lambda$ is an eigenvalue of the perturbed operator if and only if there exists a non-trivial pair $(V_s^0, V_u^0) \in E^s_+ \times E^u_-$ such that

$$\iota(V_s^0, V_u^0; \lambda, B) = 0.$$ 

Let $Q$ be a projection in $\mathbb{R}^{2n}$ onto 

$\text{Ran}\iota(\cdot, \cdot; \lambda_0, 0) = \Phi(0, T; \lambda_0, 0)E^s_+ + \Phi(0, -T; \lambda_0, 0)E^u_-$. Then

$$Q\iota(V_s^0, V_u^0, \lambda, B) = 0,$$

$$(I-Q)\iota(V_s^0, V_u^0, \lambda, B) = 0.$$
6. Main result

We show that

- \( \text{codim}(\Phi(0, T; \lambda_0, 0)E_s^+ + \Phi(0, -T; \lambda_0, 0)E_u^-) = 2m + 1 \)
- for \((\lambda, B)\) close to \((\lambda_0, 0)\), \(Q_\iota(V_0^s, V_0^u, \lambda, B) = 0\) has a unique solution

\[
(V_0^s, V_0^u) = (V_0^s(\lambda, B), V_0^u(\lambda, B))
\]

- \((I - Q)_\iota(V_0^s(\lambda, B), V_0^u(\lambda, B); \lambda, B) = 0\), gives us a smooth function \(\lambda(B)\) in a neighbourhood of \(B = 0\) such that \(\lambda(0) = \lambda_0\).
6. Main result

By means of the implicit function theorem, through Lyapunov Schmidt reduction, we now have $V_0^s, V_0^u$ and $\lambda$ as smooth functions of $B$ in a neighbourhood around $0 \in X_\beta$.

We can now prove our main theorem.
6. Main results

It is shown that $\dim(\ker Q^*) = \dim(\ker Q) = 2m + 1$, and $Z_*(0) \in \ker Q^*$, with $Z_*(0) = (-u'_*(0), u_*(0))^T$, and $u_*$ is the eigenfunction of $\mathcal{L} + B$.

We set $W_0(0) = Z_*(0)$, and further define $W_k(0) \in \mathbb{R}^{2n}$ for $k = 1, ..., 2m$, with, such that $\{W_k(0); k = 0, ..., 2m\}$ is a basis for $\ker Q^*$.

Let $W_k$ be the solution of the adjoint transformed unperturbed system, i.e., $W' = -(R(\lambda_0) + S(x; \lambda_0, 0))^*W$, with initial value $W_k(0)$. 
6. Main result

Further, define $F_k : X_\beta \to \mathbb{R}$ by

$$F_k(B) = \langle W_k(0), F(\lambda(B), B) \rangle, \quad k = 1, \ldots, 2m,$$

with $F(\lambda, B) = \iota(V_0^s(\lambda(B), B), V_0^u(\lambda(B), B); \lambda(B), B)$.

If $F_k(B) = 0$ for some $B \in X_\beta$ for all $k = 1, \ldots, 2m$, then $F(\lambda(B), B) = 0$ since $\{W_k(0); k = 0, \ldots, 2m\}$ is a basis for $\ker Q^*$. The converse clearly holds as well.

Additionally, it can be shown that $F'_k(0)$ are all linearly independent.
6. Main result

Consider the decomposition $X_\beta = \ker \overline{F}'(0) \oplus X$, where $\overline{F}(B) = (F_1(B), ..., F_{2m}(B))^T$ and $X$ has dimension $2m$.

Then for all $B \in X_\beta$, we have $B = B_1 + B_2$, for $B_1 \in \ker \overline{F}'(0)$ and $B_2 \in X$.

Define the function $f : \ker \overline{F}'(0) \times X \to \mathbb{R}^{2m}$ by

$$f(B_1, B_2) = \overline{F}(B_1 + B_2) = \overline{F}(B).$$

Differentiating gives $\partial_{B_i} f(B_1, B_2)B_i = \overline{F}'(B_1 + B_2)B_i$ which implies that $\ker \partial_{B_1} f(0, 0) = \ker \overline{F}'(0)$ and $\ker \partial_{B_2} f(0, 0) = \{0\}$. 
6. Main result

By the implicit function theorem, \( f(B_1, B_2) = 0 \) defines \( B_2 \) as a smooth function, \( g : U \subset \ker \bar{F}'(0) \rightarrow X \), of \( B_1 \) in a neighbourhood of \( B_1 = 0, B_2 = 0 \), where \( U \) is a neighbourhood of \( 0 \in \ker \bar{F}'(0) \).

Then \( g(B_1) = B_2 \) if and only if \( f(B_1, B_2) = 0 \), or equivalently, if and only if \( \bar{F}(B) = 0 \).

Further, let \( \zeta : U \subset \ker \bar{F}'(0) \rightarrow \ker \bar{F}'(0) \times X \) be defined by \( B_1 \mapsto (B_1, f(B_1)) \).

Defining \( G(x, y) = y - \zeta(x) \), we can apply the implicit function theorem again, giving the function \( h(x) = y \), defined locally. This must then be \( \zeta^{-1} \), and we are done.
We choose

$$A(x) = \begin{bmatrix} V_1(x) & 0 \\ 0 & V_2(x) \end{bmatrix},$$

with $V_1(x)$ as in the last example and $V_2(x) = 0$. We choose $\lambda_0$ as in the last example, so that for the first equation, there is a corresponding eigenfunction, $u_1$, as before. The spectrum for the second equation is purely continuous and covers, in particular, $\lambda_0$. That means that $\lambda_0$ is an embedded eigenvalue.

The corresponding eigenfunction for the system is $(u_1, 0)^T$. 
The function $F_k(B)$ can be equivalently written as

$$F_k(B) = - \int_{-\infty}^{\infty} \langle z_k(\xi), (B - (\lambda(B) - \lambda_0)I)u(\xi; B) \rangle d\xi,$$

with $z_k$ as $(G^*)^{-1}W_k = Z_k = (-z'_k, z_k)^T$. Giving

$$F'_k(0)B = \int_{-\infty}^{\infty} \langle z_k, B(\xi)u_*(\xi) \rangle d\xi - \int_{-\infty}^{\infty} \langle u_*(\xi), B(\xi)u_*(\xi) \rangle d\xi - \int_{-\infty}^{\infty} \langle z_k(\xi), u_*(\xi) \rangle d\xi$$
7. Example

In this example, for \( z_1 \) and \( z_2 \), we choose some generalized eigenfunctions

\[
\begin{align*}
  z_1(x) &= \begin{bmatrix} 0 \\ z_1(x) \end{bmatrix} \quad \text{and} \quad z_2(x) &= \begin{bmatrix} 0 \\ z_2(x) \end{bmatrix}.
\end{align*}
\]

Since

\[
B = \begin{bmatrix} b_{11}(x) & b_{12}(x) \\ b_{12}(x) & b_{22}(x) \end{bmatrix}
\]

we have

\[
\langle z_k(\xi), B(\xi)u_*(\xi) \rangle = \begin{bmatrix} 0 & z_k(\xi) \end{bmatrix} \begin{bmatrix} b_{11}(\xi) & b_{12}(\xi) \\ b_{12}(\xi) & b_{22}(\xi) \end{bmatrix} \begin{bmatrix} u_1(\xi) \\ 0 \end{bmatrix} = b_{12}(\xi) z_k(\xi) u_1(\xi),
\]
7. Example

It follows that

\[ F_k'(0)B = \int_{-\infty}^{\infty} b_{12}(\xi)z_k(\xi)u_1(\xi)d\xi \quad \text{for } k = 1, 2. \]

Hence, the manifold \( \mathcal{M} \) is tangent to the subspace of perturbations \( B \in X_\beta \) such that the off-diagonal elements are orthogonal to \( z_k(x)u_k(x) \). This follows since \( \mathcal{M} \) is described in a neighbourhood of \( B = 0 \) by the equations \( F_k(B) = 0 \), and that the eigenvalue can only persist if \( B \in \mathcal{M} \).
Thank you all for listening!