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Boundary and Eigenvalue Problems

1st Exercise Sheet

1) Find the general solution of the following differential equations:

a) $u'''(t) - u(t) = t - 1$

b) $u''(t) = 2u(t)u'(t)$

c) $\begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}.$

Hint for b): Assume first that there is a function p such that $u'(t) = p(u(t))$.

2) Let $a_0, \dots, a_n, r \in C[a,b], a_n(x) > 0$ $(x \in [a,b])$.
Show that the solution set of the n^{th} order inhomogeneous linear ODE

$$\sum_{i=0}^{n} a_i(x) u^{(i)}(x) = r(x) \qquad (x \in [a,b])$$

is given by $\{u_s + v \mid v \in V\}$ where u_s is a solution of the inhomogeneous problem and

$$V = \{\sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R}\}$$

with linearly independent solutions $v_1, \dots, v_n$ of the homogeneous equation.

3) Consider the equation of motion of a mathematical pendulum ($\omega > 0$):

$$\varphi''(t) + \omega^2 \sin \varphi(t) = 0 \qquad (*)$$

a) Prove the identity

$$[\varphi'(t)]^2 - 2\omega^2 \cos \varphi(t) = [\varphi'(0)]^2 - 2\omega^2 \quad t \in [0, \infty)$$

for any solution $\varphi : [0,\infty) \to \mathbb{R}$ of $(*)$ such that $\varphi(0) = 0$.

b) Assume that $\varphi(0) = 0,\ \varphi'(0) > 2\omega$. Show that the motion of the pendulum is periodic in the sense that $\varphi(t + T) = \varphi(t) + 2\pi$ with T given by

$$T = \frac{1}{\varphi'(0)} \int_0^{2\pi} \frac{dy}{\sqrt{1 - \frac{4\omega^2}{[\varphi'(0)]^2} \sin^2(\frac{y}{2})}}.$$

c) Describe the motion of the pendulum corresponding to the initial condition $\varphi(0) = 0, \varphi'(0) = 2\omega$ (respectively $\varphi(0) = 0, \varphi'(0) < 2\omega$).