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Boundary and Eigenvalue Problems

10th Exercise Sheet

29) A function $u \in H_0^2(\Omega)$ is called a weak solution of

$$(*) \qquad \begin{cases} \Delta^2 u = f & \text{on } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

provided that

$$\int_\Omega \Delta u \Delta v dx = \int_\Omega f v dx$$

for all $v \in H_0^2(\Omega)$. Prove that $(*)$ admits a unique weak solution for any $f \in L^2(\Omega)$.

Hint: Use without proof that

$$\inf \left\{ \frac{\|\Delta u\|_2}{\|u\|_{2,2}} : u \in H_0^2(\Omega),\ u \neq 0 \right\} > 0.$$

30) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Give a weak formulation of

$$(*) \qquad \begin{cases} -\Delta u = f & \text{on } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

and prove that a weak solution exists if and only if $\int_\Omega f dx = 0$ holds.

Hint: To prove existence, consider the usual bilinear form B on the Hilbert space

$$H = \{u \in H^1(\Omega) : \int_\Omega u dx = 0\}$$

with norm $\|u\|_H^2 = \int_\Omega |\nabla u|^2 dx$. You may use the following Poincaré inequality:

$$\left\| u - \frac{1}{|\Omega|} \int_\Omega u dx \right\|_2 \leq C \|\nabla u\|_2 \quad (u \in H^1(\Omega)).$$

31) Let $(H, \langle \cdot, \cdot \rangle)$ be a separable inner product space and let $(\phi_n)_{n \in \mathbb{N}}$ be an orthonormal system in H. Show that the following statements are equivalent:

(a) $(\phi_n)_{n \in \mathbb{N}}$ is a complete orthonormal system.
(b) For all $u \in H$: $\|u\|^2 = \sum_{i=1}^\infty |\langle u, \phi_n \rangle|^2$.
(c) For all $u, v \in H$: $\langle u, v \rangle = \sum_{i=1}^\infty \langle u, \phi_n \rangle \overline{\langle v, \phi_n \rangle}$.
(d) For all $u \in H$: If $\langle u, \phi_n \rangle = 0$ for all $n \in \mathbb{N}$ then $u = 0$.