

# Universität Karlsruhe

## Institut für Analysis

Prof. Dr. Michael Plum

Dipl.-phys. Vu Hoang

### Boundary and Eigenvalue Problems

#### 11th Exercise Sheet

**32)** Let  $R, S$  be Banach spaces and denote by  $\mathcal{B}(R, S)$  and  $\mathcal{K}(R, S)$  the spaces of all bounded linear resp. all compact linear operators from  $R$  to  $S$ , each of them endowed with the usual operator norm  $\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$  ( $A \in \mathcal{B}(R, S)$ , resp.  $A \in \mathcal{K}(R, S)$ ). Show:

- (a)  $\mathcal{B}(R, S)$  is a Banach space.
- (b)  $\mathcal{K}(R, S)$  is a closed subspace of  $\mathcal{B}(R, S)$ .

**33)** Let  $k \in L_2((0, 1) \times (0, 1))$  and

$$(Ku)(x) := \int_0^1 k(x, y)u(y) dy \quad (u \in L_2(0, 1), x \in (0, 1)).$$

Show that  $K : L_2(0, 1) \rightarrow L_2(0, 1)$  is compact.

*Hint:* Approximate  $k$  by step functions  $k_N$  (i.e. piecewise constant functions). Show that the operators

$$(K_N u)(x) := \int_0^1 k_N(x, y)u(y) dy \quad (u \in L_2(0, 1), x \in (0, 1), N \in \mathbb{N}).$$

are compact.

**34)** Prove that  $L^\infty(a, b)$  is not separable.

**35)** Let  $(\varphi_k)_{k \in \mathbb{N}}$  and  $(\psi_j)_{j \in \mathbb{N}}$  be complete orthonormal systems in  $L^2(0, 1)$ . Define  $\omega_{kj}$  by

$$\omega_{kj}(x, y) = \varphi_k(x)\psi_j(y).$$

Show that  $(\omega_{kj})_{k, j \in \mathbb{N}}$  is a complete orthonormal system in  $L^2((0, 1) \times (0, 1))$ .