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Boundary and Eigenvalue Problems

2nd Exercise Sheet

4) Let $u \in C[0,1] \cap C^2(0,1)$ satisfy the differential inequality

$$L[u] = -a(x)u''(x) + b(x)u'(x) \leq 0$$

on $[0,1]$, where $a, b \in C[0,1],\ a > 0$. Show that $u(x) \leq \max\{u(0), u(1)\}$ $(x \in (0,1))$.
Hint: Consider the function $v = u + \varepsilon e^{\lambda x}$ and show that $L[v] < 0$ for any $\varepsilon > 0$, if λ is large enough. Argue then by contradiction, assuming v has a local maximum in $(0,1)$.

5) Let

$$L[u] = \sum_{i=0}^{n} \sum_{\alpha_1 + \cdots + \alpha_m = i} a_{(\alpha_1, \ldots, \alpha_m)}(x) \frac{\partial^i u}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}$$

be a linear differential operator which is elliptic in some $x \in \mathbb{R}^m$. Prove: n is even or $m = 1$.

6) a) Prove that the operator $L[u] = -\sum_{i,j=1}^{m} \hat{a}_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$ is elliptic in $x_0 \in \mathbb{R}^m$ if and only if $\hat{A}(x_0) = (\hat{a}_{ij}(x_0))$ $(i, j = 1, \ldots, m)$ is positive or negative definite.
Which condition on $\hat{A}$ has to be imposed in order to make L uniformly strongly elliptic?

b) Let $A = (a_{ij})_{i,j=1,\ldots,m}$ be a regular x-independent matrix.
Prove that

$$L[u] = -\sum_{j=1}^{m} \left(\sum_{j=1}^{m} a_{ij} \frac{\partial}{\partial x_i} \right)^2 u$$

is a uniformly elliptic second-order differential operator (with constant coefficients).