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Boundary and Eigenvalue Problems

4th Exercise Sheet

10) Consider the homogeneous linear boundary value problem $L[u] = 0,\ U_i[u] = 0\ (i = 1, \dots, n)$, where

$$L[u](x) = \sum_{i=0}^{n} a_i(x)u^i(x) \qquad (a_i \in C[a,b])$$

and U_i are the linear boundary operators defined in the lecture. Show: If the above problem has only the trivial solution, then there is a fundamental system $\varphi_1, \dots, \varphi_n$ of the differential equation such that

$$U_i[\varphi_k] = \delta_{ik} \quad (i, k = 1, \dots, n).$$

11) Consider Sturm's boundary value problem

$$(\,*\,) \qquad \begin{cases} -(pu')' + qu = r \quad \text{on } [0,1] \\ -\alpha_0 u'(0) + \gamma_0 u(0) = \rho_0 \\ \alpha_1 u'(1) + \gamma_1 u(1) = \rho_1 \end{cases}$$

with $p \in C^1[0,1],\ q \in C[0,1],\ p(x) > 0,\ q(x) \geq 0$ on $[0,1]$ and $\alpha_i, \gamma_i \geq 0,\ \alpha_i^2 + \gamma_i^2 > 0\ \ (i = 0, 1)$.

a) Prove: If $q(x) \neq 0$ holds for some $x \in [0,1]$, or if $\gamma_0 + \gamma_1 > 0$, then the above BVP has exactly one solution for all $\rho_0, \rho_1 \in \mathbb{R}$.

b) Discuss the solvability of $(*)$ when $q \equiv 0,\ \gamma_0 = \gamma_1 = 0$.

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12) Let $f : [0,a] \times \mathbb{R} \to \mathbb{R}$ be a bounded measurable function such that, for some $h \in L^\infty(0,a)$,

$$|f(t,y) - f(t,\tilde{y})| \leq h(t)|y - \tilde{y}|$$

holds for all $y, \tilde{y} \in \mathbb{R}$ and almost all $t \in [0,a]$. $u \in L^1(0,a)$ is called a **Carathéodory** solution of the IVP $y' = f(t,y),\ y(0) = y_0$ if

$$u(x) = y_0 + \int_0^x f(t,u(t))dt \quad (x \in [0,a])$$

holds. Prove the unique solvability of the above problem by considering the operator

$$T : \begin{cases} X \to X \\ Tu(x) = y_0 + \int_0^x f(t,u(t))dt \end{cases}$$

on $X = L^1((0,a), e^{-\gamma x}dx)$ with norm

$$\|u\|_X = \int_0^a e^{-\gamma x}|u(x)|dx$$

and suitably choosen $\gamma > 0$.

Solve the IVP for $f(t,y) = \begin{cases} 0 & t \in [0,1] \\ y & t \in (1,2] \end{cases}, y_0 = 1.$

Hint: Use the integration-by-parts formula

$$\int_0^a w'(x)\left(\int_0^x v(t)dt\right)dx = -\int_0^a w(x)v(x)dx + \left[w(x)\int_0^x v(t)dt\right]_{x=0}^{x=a}$$

valid for $w \in C^1[0,a],\ v \in L^1(0,a)$.