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Boundary and Eigenvalue Problems
5th Exercise Sheet

13) Let

$$L[u](x) = a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x)$$

where $a_i \in C[a,b]\,(i = 0,1,2),\ a_2(x) \neq 0\,(x \in [a,b])$. Consider separated boundary operators

$$U_1[u] = -\alpha_0 u'(a) + \gamma_0 u(a), U_2[u] = \alpha_1 u'(b) + \gamma_1 u(b)$$

with $\alpha_0^2 + \gamma_0^2 > 0, \alpha_1^2 + \gamma_1^2 > 0$. Assume that the homogeneous problem boundary value problem only admits the trivial solution and let (φ, ψ) be fundamental system of $L[u] = 0$ such that $U_1[\varphi] = 0, U_2[\varphi] = 1, U_1[\psi] = 1, U_2[\psi] = 0$ and denote further by

$$W(x) = \det \begin{pmatrix} \varphi(x) & \psi(x) \\ \varphi'(x) & \psi'(x) \end{pmatrix}$$

the Wronskian of (φ, ψ). Prove that Green's function for (L, U_1, U_2) reads

$$G(x,t) = \frac{1}{a_2(t)W(t)} \begin{cases} \varphi(x)\psi(t) & a \leq x \leq t \leq b \\ \varphi(t)\psi(x) & a \leq t \leq x \leq b \end{cases}.$$

14) Calculate Green's function for the following problems:

a) $L[u] = -u'' + u, u(0) = u(1), u'(0) = u'(1)$

b) $L[u] = u^{(4)}, u(0) = u(1) = u'(0) = u'(1) = 0$

c) $L[u] = -u'' + c^2 u, u(0) = u(1), u'(0) = u'(1) \quad (c \neq 0)$

15) Show with the help of Green's function that the solution of the boundary value problem

$$-u''(x) = r(x) \quad (x \in (0,1)),\ u(0) = u(1) = 0$$

$(r \in C[0,1])$ satisfies

$$||u||_\infty \leq \frac{1}{8}||r||_\infty, \quad ||u'||_\infty \leq \frac{1}{2}||r||_\infty.$$