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# Boundary and Eigenvalue Problems

7th Exercise Sheet

**19)** (a) Let $\Omega = (-1,1)^2 \subset \mathbb{R}^2, u(x_1,x_2) = \begin{cases} 0 & : \quad x_1 \in (-1,0), x_2 \in (-1,1) \\ 1 & : \quad x_1 \in (0,1), x_2 \in (-1,1) \end{cases}$. Show that $\frac{\partial u}{\partial x_2}$ exists in the weak sense, but $\frac{\partial u}{\partial x_1}$ does not. Discuss what happens when $u$ is considered as a function on $\Omega_2 = (0,1) \times (-1,1)$.

(b) Calculate the derivative of $u(x_1,x_2) = |x_1| + |x_2|$ on $\mathbb{R}^2$.

(c) Let $\operatorname{sgn}(x) = 1 (x \geq 0), \operatorname{sgn}(x) = -1 (x < 0)$,

$$u(x_1,x_2) = \operatorname{sgn}(x_1) + \operatorname{sgn}(x_2)$$

on $\mathbb{R}^2$. Prove that although $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}$ do not exist in the weak sense, $\frac{\partial^2 u}{\partial x_1 \partial x_2}$ does.

**20)** Give a weak formulation of the boundary value problem

$$\begin{cases} -(au')' = 0 & x \in (-1,1) \\ u(-1) = 3, u(1) = 0 & \end{cases}$$

where $a(x) = 1$ on $(-1,0)$ and $a(x) = \frac{1}{2}$ on $[0,1)$. Determine its solution.

**21)** Let $\Omega \subset \mathbb{R}^n$ an open set, $A \subset \mathbb{R}^m$ a measurable set and $f : \Omega \times A \to \mathbb{R}$ such that $f \in L^1(\Omega \times A)$. Moreover, assume that for almost all $y \in A$ and some $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $D^\alpha f(\cdot, y)$ exists in the weak sense on $\Omega$ and $D^\alpha f(\cdot,\cdot) \in L^1(\Omega \times A)$. Show that

$$D^\alpha \left( \int_A f(\cdot, y) dy \right) = \int_A D^\alpha f(\cdot, y) dy$$

on $\Omega$ in the sense of weak differentiability.

*Hint:* Use Fubini's theorem.

**22)** Prove that the spaces $H^{k,p}(\Omega)$ defined in the lecture are Banach spaces. As a known fact, use the completeness of $L^p(\Omega)$.