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Boundary and Eigenvalue Problems

8th Exercise Sheet

23) Prove that $H^1(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$. You may use the fact that any $u \in H^1(\mathbb{R}^n)$ may be approximated in H^1-norm by functions in $L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

Hint: There exists a $\zeta \in C_0^\infty(\mathbb{R}^n)$ such that $\zeta(x) = 0$ for $|x| > 2$ and $\zeta(x) = 1$ for $|x| \leq 1$. Consider the functions $u_{kj}(x) = \zeta(\frac{x}{k})u_j(x)$ where u_j converges in H^1 to u.

24) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1-boundary. Prove the existence of a $f \in C^1(\bar{\Omega}, \mathbb{R}^n)$ such that $f \cdot \nu \geq 1$ on $\partial\Omega$, where ν denotes the outward-pointing unit normal.

Hint: Let $x_0 \in \partial\Omega$. In a neighbourhood of x_0, $\partial\Omega$ can be represented as the graph of a C^1-function. Construct f in such a neighbourhood and apply the following theorem (partition of unity) to define f on Ω:

Theorem: Let $K \subset \mathbb{R}^n$ be compact and $V_1, \ldots, V_m \subset \mathbb{R}^n$ open sets which have a nonempty intersection with K such that

$$K \subset \bigcup_{i=1}^{m} V_i.$$

Then there exist $\varphi_i \in C_0^\infty(\mathbb{R}^n)$ such that $\text{supp}\varphi \subset V_i, 0 \leq \varphi_i \leq 1$ and $\varphi_1 + \ldots + \varphi_m = 1$ on K.

25) Let $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Show that $H^{k-\frac{1}{p},p}(\partial\Omega)$ is a normed space and prove moreover its completeness for $p = 2$.

Hint for completeness: Consider for $g \in H^{\frac{1}{2},2}$ the boundary value problem

$$(*) \quad \int_\Omega (\nabla u \cdot \nabla\varphi + u\varphi)dx = 0 \quad (\varphi \in H_0^1(\Omega)), \quad u|_{\partial\Omega} = g.$$

Prove the unique solvability of (*) and define a bounded operator $A : H^{\frac{1}{2},2}(\Omega) \to H^{1,2}(\Omega)$ by $A[g] := u^*$ where u^* is the solution to (*).