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Boundary and Eigenvalue Problems

9th Exercise Sheet

26) a) Let $u \in H^1(0,1)$. Prove that there exists a representative of (the equivalence class) u in $C[0,1]$.
Hint: Derive first for $u \in C^1[0,1]$ the inequality

$$|u(x) - u(y)| \leq |x-y|^{\frac{1}{2}} ||u'||_2 \quad (x, y \in (0,1))$$

b) Let $u \in H^1((0,1)^2)$. Show that $u(\cdot, x_2) \in H^1(0,1)$ for almost all $x_2 \in (0,1)$.

27) Let $u \in H_0^1(0,1)$ be a weak solution of

$$-u'' + bu' + cu = f$$

on $(0,1)$, where $b, c \in L^\infty(0,1), f \in L^2(0,1)$. Show that $u \in H^2(0,1)$ and that u satisfies the equation $-u''(x) + b(x)u'(x) + c(x)u(x) = f(x)$ in the pointwise sense almost everywhere on $(0,1)$.

Hint: Prove the following first: if there exists a $C > 0$ such that

$$\left| \int_0^1 u' \varphi' dx \right| \leq C ||\varphi||_2 \quad (\varphi \in H_0^1(0,1)),$$

then $u \in H^2(0,1)$.

28) Let $h(x) := 0$ for $x \in (-\pi, 0)$ and $h(x) := 1$ for $x \in (0, \pi)$. Determine all $\lambda \in \mathbb{R}$ for which the following weakly formulated boundary value problem

$$\int_{-\pi}^{\pi} [u'(x)\varphi'(x) - h(x)(u\varphi)'(x)] dx = \lambda \int_{-\pi}^{\pi} u(x)\varphi(x) dx$$

$(\varphi \in H_0^1(-\pi, \pi))$ admits a nontrivial solution $u \in H_0^1(-\pi, \pi)$. If you have a background in quantum mechanics, explain the connection between the above problem and the quantum-mechanical problem of a particle trapped in a box divided into two parts by an infinitely thin wall.