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SS 2020
29.04.2020

Boundary and Eigenvalue Problems

Exercise sheet 1

Exercise 1

Determine all eigenvalues $\lambda \in \mathbb{C}$ and eigenfunctions of the boundary value problem

$$-v''(x) + v'(x) = \lambda v(x), \quad x \in [0, \pi], \quad v(0) = v(\pi), \quad v'(0) = v'(\pi).$$

Exercise 2

Let $r(t) := \sin(\tilde{\omega} t)$, $\tilde{\omega} \in \mathbb{R}$ fixed. Determine for all $\gamma, \omega \in \mathbb{R}$ the general solution of the harmonic oscillator with friction

$$u^{''} + \gamma u' + \omega^2 u = r.$$

Exercise 3

Consider the equation of the motion of a mathematical pendulum ($\omega > 0$):

$$\varphi''(t) + \omega^2 \sin \varphi(t) = 0. \tag{1}$$

a) Prove the identity

$$[\varphi'(t)]^2 - 2\omega^2 \cos \varphi(t) = [\varphi'(0)]^2 - 2\omega^2, \quad t \in [0, \infty)$$

for any solution $\varphi : [0, \infty) \to \mathbb{R}$ of (1) such that $\varphi(0) = 0$.

b) Assume that $\varphi(0) = 0,\ \varphi'(0) > 2\omega$. Show that the motion of the pendulum is periodic in the sense that $\varphi(t + T) = \varphi(t) + 2\pi$ with T given by

$$T = \frac{1}{\varphi'(0)} \int_0^{2\pi} \frac{dy}{\sqrt{1 - \frac{4\omega^2}{[\varphi'(0)]^2} \sin^2(\frac{y}{2})}}.$$

c) Describe the motion of the pendulum corresponding to the initial condition $\varphi(0) = 0, \varphi'(0) = 2\omega$ (respectively $\varphi(0) = 0, \varphi'(0) < 2\omega$).