Exercise 28

Let $H$ be a Hilbert space and let $A : D(A) \to H$ be a densely defined linear operator. Show that $A^*$ is closed.

Exercise 29

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $A : D(A) \to H$ be a densely defined symmetric linear operator. Show that the following statements are pairwise equivalent:

(i) $A$ is self-adjoint.

(ii) $A$ is closed and $\ker (A^* \pm i) = \{0\}$.

(iii) $(A \pm i)(H) = H$.

Furthermore, the following statements are pairwise equivalent:

(i) $A$ is essentially self-adjoint.

(ii) $\ker (A^* \pm i) = \{0\}$.

(iii) $(A \pm i)(H)$ is dense in $H$.

Hint: Prove the following lemma first.

Lemma 1. Let $A : D(A) \to H$ be linear and densely defined. Then the following statements hold:

(i) $\ker (A^* \mp i) = \left[ (A \pm i)(H) \right]^\perp$.

(ii) Let $A$ be closed and symmetric. Then $(A \mp i)(H)$ is closed.

Exercise 30

(i) Let $v \in L^2(0, 1)$ and $u := \int_0^1 v(t) \, dt$. Show that $u \in H^1_{(0)}(0, 1) := C^\infty_0((0, 1]) \| \cdot \|_{H^1}$ and $u' = v$.

Hint: Approximate $v$ by a sequence in $C^\infty_0((0, 1])$.

(ii) Consider the linear operator $A : D(A) \to L^2(0, 1)$ defined by $D(A) := H^1_0(0, 1) \subset L^2(0, 1)$ and $Au = iu'$. Discuss whether $A$ is closed, symmetric or self-adjoint. Which of these properties continue to hold when the domain of the operator is replaced by $H^1_0(0, 1)$?