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Boundary and Eigenvalue Problems

Exercise sheet 10

Exercise 28

Let H be a Hilbert space and let $A : D(A) \to H$ be a densely defined linear operator. Show that A^* is closed.

Exercise 29

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $A : D(A) \to H$ be a densely defined symmetric linear operator. Show that the following statements are pairwise equivalent:

(i) A is self-adjoint.

(ii) A is closed and $\ker(A^* \pm i) = \{0\}$.

(iii) $(A \pm i)(H) = H$.

Furthermore, the following statements are pairwise equivalent:

(i) A is essentially self-adjoint.

(ii) $\ker(A^* \pm i) = \{0\}$.

(iii) $(A \pm i)(H)$ is dense in H.

Hint: Prove the following lemma first.

Lemma 1. *Let $A : D(A) \to H$ be linear and densely defined. Then the following statements hold:*

(i) $\ker(A^* \mp i) = \big[(A \pm i)(H)\big]^\perp$.

(ii) Let A be closed and symmetric. Then $(A \pm i)(H)$ is closed.

Exercise 30

(i) Let $v \in L^2(0,1)$ and $u := \int_0^\cdot v(t)\,dt$. Show that $u \in H^1_{\{0\}}(0,1) := \overline{C_0^\infty((0,1])}^{\|\cdot\|_{H^1}}$ and $u' = v$.

Hint: Approximate v by a sequence in $C_0^\infty((0,1])$.

(ii) Consider the linear operator $A\colon D(A) \to L^2(0,1)$ defined by $D(A) := H^1_{\{0\}}(0,1) \subset L^2(0,1)$ and $Au = iu'$. Discuss whether A is closed, symmetric or self-adjoint. Which of these properties continue to hold when the domain of the operator is replaced by $H^1_0(0,1)$?