Exercise 31

Let $H$ be a Hilbert space. Let $A: D(A) \to H$ be a densely defined, positive semi-definite, self-adjoint linear operator. Show that $A - \lambda$ is surjective for all $\lambda < 0$.

Exercise 32

Let $D(A) = H^2(\mathbb{R}^n)$. Let $A: D(A) \to L^2(\mathbb{R}^n)$ and $Au := -\Delta u$. Show that $\sigma(A) = [0, \infty)$.

Hint: Use without proof that $A$ is self-adjoint.

Exercise 33

Prove or provide a counterexample for each of the following statements:

(i) If $u \in H^1(\mathbb{R})$, then $u(x) \to 0$ as $|x| \to \infty$.

Hint: Notice that $u(x) = \int_{x-1}^{x} \frac{\partial((t-x+1)u(t))}{\partial t} dt$ for almost all $x \in \mathbb{R}$.

(ii) If $u \in L^2(\mathbb{R})$, then $u(x) \to 0$ as $|x| \to \infty$.

Exercise 34

Let $H = L^2(\mathbb{R})$. The Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is given by \( \langle u, v \rangle = \int_{\mathbb{R}} u(x)\overline{v(x)} \, dx \).

Define $D(B) = \{ u \in L^2(\mathbb{R}): \int_{\mathbb{R}} |x||u(x)|^2 \, dx < \infty\}$. Let $B[u,v]: D(B) \times D(B) \to \mathbb{C}$ be defined by

$$B[u,v] = \int_{\mathbb{R}} |x|u(x)v(x) \, dx.$$ 

Show that there is a self-adjoint operator $A$ such that $B[u,v] = \langle Au, v \rangle$ for all $u \in D(A)$ and $v \in D(B)$. State the explicit domain $D(A)$. 