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Boundary and Eigenvalue Problems

Exercise sheet 11

Exercise 31

Let H be a Hilbert space. Let $A\colon D(A) \to H$ be a densely defined, positive semi-definite, self-adjoint linear operator. Show that $A - \lambda$ is surjective for all $\lambda < 0$.

Exercise 32

Let $D(A) = H^2(\mathbb{R}^n)$. Let $A\colon D(A) \to L^2(\mathbb{R}^n)$ and $Au \coloneqq -\Delta u$. Show that $\sigma(A) = [0, \infty)$.
Hint: Use without proof that A is self-adjoint.

Exercise 33

Prove or provide a counterexample for each of the following statements:

(i) If $u \in H^1(\mathbb{R})$, then $u(x) \to 0$ as $|x| \to \infty$.
Hint: Notice that $u(x) = \int_{x-1}^{x} \frac{\partial[(t-x+1)u(t)]}{\partial t}\, dt$ for almost all $x \in \mathbb{R}$.

(ii) If $u \in L^2(\mathbb{R})$, then $u(x) \to 0$ as $|x| \to \infty$.

Exercise 34

Let $H = L^2(\mathbb{R})$. The Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is given by $\langle u, v\rangle = \int_{\mathbb{R}} u(x)\overline{v(x)}\, dx$.
Define $D(B) = \{u \in L^2(\mathbb{R})\colon \int_{\mathbb{R}} |x||u(x)|^2\, dx < \infty\}$. Let $B[u, v]\colon D(B) \times D(B) \to \mathbb{C}$ be defined by

$$B[u, v] = \int_{\mathbb{R}} |x|u(x)\overline{v(x)}\, dx.$$

Show that there is a self-adjoint operator A such that $B[u, v] = \langle Au, v\rangle$ for all $u \in D(A)$ and $v \in D(B)$. State the explicit domain $D(A)$.