Exercise 35 (Neumann series theorem.)

Let \( B : H \to H \) be a bounded linear operator with \( \| B \| < 1 \). Show that \( I - B \) has a bounded inverse \( (I - B)^{-1} : H \to H \) with \( \|(I - B)^{-1}\| \leq 1/(1 - \| B \|) \).

**Hint:** Banach fixed point theorem.

Exercise 36

Let \( H \) be a Hilbert space. Let \( A : D(A) \to H \), \( D(A) \subseteq H \), be a self-adjoint operator satisfying \( \langle Au, u \rangle \geq c \langle u, u \rangle \) for some \( c \in \mathbb{R} \) and for all \( u \in D(A) \). Let \( V \) be a symmetric operator with \( D(A) \subseteq D(V) \). Show that if there exist \( a, b \in \mathbb{R} \) such that \( 0 \leq a < 1, b \geq 0 \) and

\[
\| V\phi \| \leq a \| A\phi \| + b \| \phi \| \quad \text{(for all } \phi \in D(A)),
\]

then \( A + V \) is self-adjoint on \( D(A) \).

**Hint:** Show that

\[
\limsup_{\lambda \to \infty} \| V(A + \lambda)^{-1} \| < 1.
\]

Use without proof that

\[
\| A(A + \lambda)^{-1} \| = \sup_{s \in \sigma(A)} \left| \frac{s}{s + \lambda} \right|.
\]

Exercise 37

Let \( H = L^2(0, \infty) \). The Hilbert space \((H, \langle \cdot, \cdot \rangle)\) is given by \( \langle u, v \rangle = \int_0^\infty u(x)\overline{v(x)} \, dx \). Let \( B[u, v] = \int_0^\infty u'(x)\overline{v'(x)} \, dx \) and \( D(B) = H^1(0, \infty) \). Show that \( B \) is a densely defined, closed, symmetric and positive semi-definite sesquilinear form. State the explicit domain \( D(A) \) of the self-adjoint operator \( A \) provided by Theorem VI.6.

Exercise 38

Let \( H \) be a Hilbert space. We say that a symmetric sesquilinear form \( B \) is closable, if there exists a closed symmetric sesquilinear form which extends \( B \). Let \( H = L^2(\mathbb{R}) \) and let \( B(u, v) = u(0)\overline{v(0)} \) be defined on \( D(B) = L^2(\mathbb{R}) \cap C^0(\mathbb{R}) \). Show that \( B \) is densely defined, symmetric and positive semi-definite but not closable.