Karlsruhe Institute of Technology (KIT) SS 2020
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Boundary and Eigenvalue Problems

Exercise sheet 12

Exercise 35 (Neumann series theorem.)

Let $B : H \to H$ be a bounded linear operator with $\|B\| < 1$. Show that $I - B$ has a bounded inverse $(I - B)^{-1} \colon H \to H$ with $\|(I - B)^{-1}\| \leq 1/(1 - \|B\|)$.

Hint: Banach fixed point theorem.

Exercise 36

Let H be a Hilbert space. Let $A : D(A) \to H$, $D(A) \subseteq H$, be a self-adjoint operator satisfying $\langle Au, u\rangle \geqslant c\langle u, u\rangle$ for some $c \in \mathbb{R}$ and for all $u \in D(A)$. Let V be a symmetric operator with $D(A) \subseteq D(V)$. Show that if there exist a, $b \in \mathbb{R}$ such that $0 \leqslant a < 1, b \geqslant 0$ and

$$\|V\phi\| \leqslant a\|A\phi\| + b\|\phi\| \quad \text{(for all } \phi \in D(A)), \tag{1}$$

then $A + V$ is self-adjoint on $D(A)$.

Hint: Show that

$$\limsup_{\lambda\to\infty} \left\|V(A + \lambda)^{-1}\right\| < 1.$$

Use without proof that

$$\left\|A(A + \lambda)^{-1}\right\| = \sup_{s\in\sigma(A)} \left|\frac{s}{s + \lambda}\right|.$$

Exercise 37

Let $H = L^2(0, \infty)$. The Hilbert space $(H, \langle\cdot, \cdot\rangle)$ is given by $\langle u, v\rangle = \int_0^\infty u(x)\overline{v(x)}\, dx$. Let $B[u, v] = \int_0^\infty u'(x)\overline{v'(x)}\, dx$ and $D(B) = H^1(0, \infty)$. Show that B is a densely defined, closed, symmetric and positive semi-definite sesquilinear form. State the explicit domain $D(A)$ of the self-adjoint operator A provided by Theorem VI.6.

Exercise 38

Let H be a Hilbert space. We say that a symmetric sesquilinear form B is closable, if there exists a closed symmetric sesquilinear form which extends B. Let $H = L^2(\mathbb{R})$ and let $B(u, v) = u(0)\overline{v(0)}$ be defined on $D(B) = L^2(\mathbb{R}) \cap C^0(\mathbb{R})$. Show that B is densely defined, symmetric and positive semi-definite but not closable.