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Boundary and Eigenvalue Problems

Exercise sheet 13

Exercise 39

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\lambda_i(\Omega)$ be the i-th eigenvalue of the Dirichlet eigenvalue problem

$$u \in H_0^1(\Omega): \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega uv \, dx, \quad \text{for all } v \in H_0^1(\Omega).$$

The sequence of eigenvalues can be ordered: $\lambda_1(\Omega) \leqslant \lambda_2(\Omega) \leqslant \ldots$.

(i) Show that if $\Omega_1 \subset \Omega_2$, then $\lambda_i(\Omega_2) \leqslant \lambda_i(\Omega_1)$ for all $i \in \mathbb{N}$.

(ii) Let $a, b > 0$ and let

$$E := \left\{ (x,y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1 \right\}$$

be an ellipse. Find the smallest interval $[\lambda_1(\Omega_2), \lambda_1(\Omega_1)]$ containing $\lambda_1(E)$, where $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ are axis-parallel rectangles and $\Omega_1 \subset E \subset \Omega_2$.

Exercise 40 (Continuous dependence of the eigenvalues on coefficients.)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain.

(i) Let $c \in L^\infty(\Omega)$ and $\lambda_i(c)$ be the i-th eigenvalue (ordered increasingly) of the eigenvalue problem

$$u \in H_0^1(\Omega): \quad \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega cuv \, dx = \lambda \int_\Omega uv \, dx, \quad \text{for all } v \in H_0^1(\Omega).$$

Show that if $c, \tilde{c} \in L^\infty(\Omega)$ then

$$|\lambda_i(c) - \lambda_i(\tilde{c})| \leqslant \|c - \tilde{c}\|_{L^\infty} \quad (i \in \mathbb{N}).$$

(ii) Let $w \in L^\infty(\Omega)$ such that $w(x) \geqslant \underline{w} > 0$ $(x \in \Omega)$ for some constant $\underline{w} > 0$. Let $\lambda_i(w)$ be i-th eigenvalue (ordered increasingly) of the eigenvalue problem

$$u \in H_0^1(\Omega): \quad \int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega wuv \, dx \quad \text{for all } v \in H_0^1(\Omega).$$

Show that if $w, \tilde{w} \in L^\infty(\Omega)$ satisfy $w(x), \tilde{w}(x) > \underline{w} > 0$ $(x \in \Omega)$ and $\left\|1 - \frac{\tilde{w}}{w}\right\|_{L^\infty} < 1$, then

$$|\lambda_i(w) - \lambda_i(\tilde{w})| \leqslant \left\|1 - \frac{\tilde{w}}{w}\right\|_{L^\infty} \lambda_i(\tilde{w}) \quad (i \in \mathbb{N}).$$