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Boundary and Eigenvalue Problems

Exercise sheet 14

Exercise 41

Consider the eigenvalue problem

$$\begin{cases} -u''(x) = \lambda(2+\sin(x))u(x) \quad \text{in} \quad (0,\pi), \\ \quad u(0) = u(\pi) = 0. \end{cases} \tag{1}$$

(i) Determine a weak formulation of the problem (1) using an appropriate bilinear form B on the space $L^2((0,\pi); g)$, where the inner product is given by $\langle u, v\rangle = \langle gu, v\rangle_{L^2}$ with $g(x) = 2 + \sin(x)$.

(ii) Use the Rayleigh–Ritz method to calculate an upper bound Λ_1 for the first eigenvalue λ_1 of the eigenvalue problem (1).

Hint: Use $\tilde{u}(x) = \sin(x)$ as a test function.

(iii) Determine a lower bound ρ for the eigenvalue λ_2 such that $\Lambda_1 < \rho$ by comparing the problem (1) with the problem

$$\begin{cases} -u''(x) = 3\mu u(x) \quad \text{in} \quad (0,\pi), \\ \quad u(0) = u(\pi) = 0. \end{cases} \tag{2}$$

(iv) Use the Temple–Lehmann method to determine a lower bound $\underline{\lambda}_1$ for λ_1.

Hint: Use $\tilde{u}(x) = \sin(x)$ as a test function.

Exercise 42

Let $\Omega \subset \mathbb{R}^d$ be a bounded C^1 domain. Let

$$D(B) = \{u \in H^1(\Omega) : \int_\Omega u\,dx = 0\}, \quad B\colon D(B)\times D(B) \to \mathbb{R}, \quad B[u,v] = \int_\Omega \nabla u \cdot \nabla v\,dx.$$

Consider the eigenvalue problem

$$B[u,v] = \lambda\langle u, v\rangle_{L^2} \quad \text{for all} \quad v \in D(B).$$

Determine the strong formulation of this problem. Furthermore, define

$$X \coloneqq D(B) \times L^2(\Omega),$$
$$T\colon D(B) \to X, \quad Tu \coloneqq (u,u),$$
$$b\colon X\times X \to \mathbb{R}, \quad b((\hat{u}_1,\hat{u}_2),(\hat{v}_1,\hat{v}_2)) \coloneqq \int_\Omega \nabla\hat{u}_1\cdot\nabla\hat{v}_1\,dx + c\int_\Omega (\hat{u}_2\hat{v}_2 - \hat{u}_1\hat{v}_1)\,dx, \quad c\in\mathbb{R}.$$

(i) Show that for an appropriate choice of $c > 0$, b is positive semi-definite and

$$B[u,v] = b(Tu,Tv) \quad (u,v\in D(B)).$$

Hint: Use without proof the following version of the Poincaré inequality:

Theorem 1 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain and let $\overline{u}_\Omega := \frac{1}{|\Omega|}\int_\Omega u\,dx$. Let $1 \leqslant p \leqslant \infty$. There exists a constant $C = C(n, p, \Omega)$ such that*

$$\|u - \overline{u}_\Omega\|_{L^p} \leqslant C\|\nabla u\|_{L^p} \quad \textit{for all } u \in W^{1,p}(\Omega).$$

(ii) For a given $\tilde{u} \in D(B)$, determine $\hat{w} \in X$ such that

$$b(\hat{w}, Tv) = \langle \tilde{u}, v\rangle_{L^2} \quad (v \in D(B)).$$

Exercise 43

Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space. Let $D(B) \subseteq H$ and $B\colon D(B) \times D(B) \to \mathbb{R}$ be a symmetric positive semi-definite bilinear form. Let the eigenfunctions of the eigenvalue problem $B[u, v] = \lambda\langle u, v\rangle$ (for all $v \in D(B)$) form an orthonormal basis $(u_i)_{i\in N}$ of H with corresponding eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ with $\lambda_i \to \infty \quad (i \to \infty)$. For a given $u \in H$, consider the following problem occurring in the Temple-Lehmann method: Find $w \in D(B)$ such that

$$B[w, v] = \langle u, v\rangle \quad \text{for all} \quad v \in D(B). \tag{3}$$

(i) Show that $w \in D(B)$ is a solution of the problem (3) if and only if

$$-B[w, w] = \min\{B[v, v] - 2\langle u, v\rangle : v \in D(B)\}.$$

Furthermore, the minimum is attained at $v = w$.

(ii) Let X be a real vector space. Let $T : D(B) \to X$ be a linear operator and $b : X \times X \to \mathbb{R}$ be a symmetric positive semi-definite bilinear form such that

$$B[\tilde{u}, \tilde{v}] = b(T\tilde{u}, T\tilde{v}) \quad (\tilde{u}, \tilde{v} \in D(B)).$$

Show that if $w \in D(B)$ is a solution of the problem (3), then

$$-B[w, w] = \max\{-b(g, g) : g \in V_u\},$$

where $V_u := \{g \in X : b(g, Tv) = \langle u, v\rangle \text{ for all } v \in D(B)\}$.