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Boundary and Eigenvalue Problems

Exercise sheet 2

Exercise 4

Let

$$L[u] = \sum_{i=0}^{n} \sum_{\substack{\alpha_1,\dots,\alpha_m \in \mathbb{N}_0 \\ \alpha_1+\dots+\alpha_m = i}} a_{(\alpha_1,\dots,\alpha_m)}(x) \frac{\partial^i u}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}$$

be a linear differential operator which is elliptic at some $x \in \mathbb{R}^m$. Prove that n is even or $m = 1$.

Exercise 5

Let

$$L[u] = -\sum_{i,j=1}^{m} \hat{a}_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u$$

be a linear second order differential operator.

a) Write L as in the lecture notes as a sum over multi-indices (see Exercise 4).

b) Prove that the operator L is elliptic at $x_0 \in \mathbb{R}^m$ if and only if $\hat{A}(x_0) = (\hat{a}_{ij}(x_0))_{i,j=1,\dots,m}$ is positive or negative definite.

c) Which condition on $\hat{A}$ has to be imposed in order to make L uniformly strongly elliptic?

Exercise 6

a) Let $L[u] = (-\Delta)^k u := \underbrace{(-\Delta) \circ \dots \circ (-\Delta)}_{k-\text{times}} u$. Show that L is uniformly strongly elliptic.

b) Let L be the quasilinear differential operator defined by $L[u] = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$. Show that L is elliptic.

c) Let $L[u] = x_1 \frac{\partial^2 u}{\partial x_1^2} + 2x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + x_1 \frac{\partial^2 u}{\partial x_2^2} + (x_1^2 + x_2^2) \frac{\partial u}{\partial x_1} + (x_1^2 x_2) \frac{\partial u}{\partial x_2} + \sin(x_1 + x_2)u$. Determine all points $x = (x_1, x_2) \in \mathbb{R}^2$, at which L is elliptic.