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Boundary and Eigenvalue Problems

Exercise sheet 3

Exercise 7

Let $v \in C^2(\mathbb{R}^3;\mathbb{R}^3)$ be the velocity field of a laminar flow, i.e. $\operatorname{curl}(v) = 0$. Suppose that a density-velocity relation of the form $\rho = f(|v|^2)$ holds, where $f \in C^1(\mathbb{R})$. Show that

1. there exists a scalar function (pressure) $p \in C^1(\mathbb{R}^3)$ such that the triple (ρ, v, p) satisfies the stationary Euler equation
$$\rho(v \cdot \nabla)v + \nabla p = 0.$$
2. if $v \in C^3(\mathbb{R}^3;\mathbb{R}^3)$, there exists a scalar function (pressure) $p \in C^1(\mathbb{R}^3)$ such that the triple (ρ, v, p) satisfies the stationary Navier–Stokes equation
$$\rho(v \cdot \nabla)v - \alpha\Delta v - \beta\nabla(\operatorname{div}(v)) + \nabla p = 0,$$
where α, $\beta \in \mathbb{R}$ are given material constants.

Exercise 8

Consider *Sturm's* boundary value problem

$$\begin{aligned} -(p(x)u'(x))' + q(x)u(x) &= r(x)\,, x \in [0,1] \\ -\alpha_0 u'(0) + \gamma_0 u(0) &= \rho_0, \\ \alpha_1 u'(1) + \gamma_1 u(1) &= \rho_1, \end{aligned}$$

where $p \in C^1([0,1])$, $q \in C([0,1])$; $p(x) > 0, q(x) \geq 0$ on $[0,1]$; $\alpha_i, \gamma_i \geq 0$ and $\alpha_i^2 + \gamma_i^2 > 0$ $(i = 1,2)$.

1. Show that if $q(x) \neq 0$ for at least one $x \in [0,1]$ or $\gamma_0 > 0$ or $\gamma_1 > 0$ then the above boundary value problem is uniquely solvable for all $r \in C([0,1])$ and all $\rho_0, \rho_1 \in \mathbb{R}$.
2. Discuss the solvability of the boundary value problem in the case $q(x) = 0$ $(x \in [0,1])$, $\gamma_0 = \gamma_1 = 0$; determine, in the solvable case, the general solution.

Exercise 9

Show that the nonlinear boundary value problem

$$y''(x) + |y(x)| = 0, \quad y(0) = 0, \quad y(\pi + \ln 2) = -1$$

admits exactly two solutions.

Hint: Consider the initial value problem

$$y''(x) + |y(x)| = 0, \quad y(0) = 0, \quad y'(0) = s, \tag{1}$$

where $s \in \mathbb{R}$. Determine for all $s \in \mathbb{R}$ the general solution $y^{(s)}$ of (1), first on $[0,\pi]$ and then on $[0, \pi + \ln 2]$. Choose $s \in \mathbb{R}$ such that $y^{(s)}(\pi + \ln 2) = -1$ (Shooting method).