Boundary and Eigenvalue Problems
Exercise sheet 4

Exercise 10

Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and consider the boundary value problem

$$\begin{cases}
 u_{xy} = 0 & \text{in } \Omega \\
 u = u_0 & \text{on } \partial \Omega 
\end{cases}$$

(1)

for $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and given $u_0 \in C(\partial \Omega)$. Prove that the relation

$$u(0,0) + u(1,1) = u(1,0) + u(0,1)$$

holds for every solution of (1) and conclude that there exists a $u_0 \in C(\partial \Omega)$ such that (1) has no solution. Is the operator $L[u] = u_{xy}$ elliptic?

Exercise 11

Let $\Omega \subset \mathbb{R}^n$ be a $C^1$-domain. Let $A \in C^1(\overline{\Omega}; \mathbb{R}^{n \times n})$, $b \in C(\Omega, \mathbb{R}^n)$, $c, r \in C(\overline{\Omega})$ and $\gamma \in C(\partial \Omega)$. Show that if $u \in C^2(\overline{\Omega})$ satisfies the weak formulation (S) of the boundary value problem

$$\begin{cases}
 - \text{div}(A \nabla u) + b \cdot \nabla u + cu = r & \text{in } \Omega \\
 u = 0 & \text{on } \Gamma_0 \\
 \frac{\partial u}{\partial n} + \gamma u = 0 & \text{on } \Gamma_1 
\end{cases}$$

and $u = 0$ on $\Gamma_0$ then $u$ is also a classical solution.

Exercise 12

Consider the boundary value problem

$$-(au')' = 1 \quad (-1 < x < 1), \quad u(-1) = u(1) = 0,$$

where $a(x) = \begin{cases} 
1 & \text{if } -1 \leq x < 0, \\
\frac{1}{2} & \text{if } 0 \leq x \leq 1. 
\end{cases}$

Determine the weak formulation of this problem and its solution.