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Boundary and Eigenvalue Problems

Exercise sheet 5

Exercise 13

Let V be a real vector space and let B be a symmetric and positive bilinear form on V, i.e.

$$B[u,v] = B[v,u] \text{ for all } u,v \in V, \quad B[u,u] > 0 \text{ for all } u \in V \setminus \{0\}.$$

Let F be a linear functional on V. Show that the nonlinear functional

$$J : \begin{cases} V & \longrightarrow & \mathbb{R} \\ u & \longmapsto & \frac{1}{2}B[u,u] - F[u] \end{cases}$$

attains its minimum at $u_0 \in V$ if and only if

$$B[u_0, \phi] = F[\phi] \text{ for all } \phi \in V.$$

Exercise 14

1. Show that the function u on $[-1,1]$ defined as follows is not weakly differentiable:

$$u(x) = \begin{cases} -1 & \text{if } -1 \le x < 0, \\ 1 & \text{if } 0 \le x \le 1. \end{cases}$$

2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Show that $C(\overline{\Omega}) \not\subseteq H^1(\Omega)$.
3. Let $\Omega \subset \mathbb{R}^2$ be open and bounded. Show that $H^1(\Omega) \not\subseteq C(\overline{\Omega})$.
   Hint: Consider the function $u(x,y) = \log|\log(\sqrt{x^2+y^2})|$.
   Does the same statement hold true for $n \ge 3$?
4. Let $\Omega = (0,1)^2$, $\Omega_1 = (0, \frac{1}{2}) \times (0,1)$ and $\Omega_2 = (\frac{1}{2}, 1) \times (0,1)$. Let the function $u : \overline{\Omega} \to \mathbb{R}$ satisfy $u|_{\overline{\Omega}_1} \in C^1(\overline{\Omega}_1)$ and $u|_{\overline{\Omega}_2} \in C^1(\overline{\Omega}_2)$. Show that $u \in H^1(\Omega)$ if and only if $u \in C(\overline{\Omega})$.

Exercise 15

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $r \in L^2(\Omega)$. Consider the boundary value problem

$$\Delta^2 u = r \text{ in } \Omega, \quad u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Determine the weak formulation of the boundary value problem and show that there exists a unique weak solution.

Hint: You can use without proof that $\inf \left\{ \frac{\|\Delta u\|_{L^2}}{\|u\|_{H^2}} : u \in H_0^2(\Omega) \setminus \{0\} \right\} > 0$.