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Boundary and Eigenvalue Problems

Exercise sheet 6

Exercise 16

Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain. Show the existence of a function $f \in C^1(\overline{\Omega}; \mathbb{R}^n)$ such that $f \cdot \nu \geqslant 1$ on $\partial\Omega$, where ν denotes the outer unit normal vector.

Hint: For any $x_0 \in \partial\Omega$ there exists a neighbourhood U of x_0, such that $U \cap \partial\Omega$ can be represented as the graph of a C^1 function in some suitable coordinate system. Construct f in such a neighbourhood then use the following theorem to define f on Ω:

Theorem (partition of unity). *Let $K \subset \mathbb{R}^n$ be compact and let $V_1, \ldots, V_m \subset \mathbb{R}^n$ be open such that*

$$K \subset \bigcup_{i=1}^{m} V_i.$$

Then there exist $\varphi_i \in C_0^\infty(\mathbb{R}^n)\,(i = 1, ..., m)$, such that supp $\varphi_i \subset V_i$, $0 \leqslant \varphi_i \leqslant 1$ *and* $\varphi_1 + \ldots + \varphi_m = 1$ *on K.*

Exercise 17

Let $p \in [1, \infty)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain.
Show that $(H^{k-\frac{1}{p},p}(\partial\Omega), ||\cdot||_{k-\frac{1}{p},p})$ is a normed vector space. Furthermore, show completeness for the case $p = 2$, $k = 1$.

Hint: To prove the completeness, consider, for given $g \in H^{\frac{1}{2},2}(\partial\Omega)$, the boundary value problem

$$\int_\Omega \nabla u \cdot \nabla \phi + u\phi \, dx = 0 \quad (\phi \in H_0^1(\Omega)), \quad u|_{\partial\Omega} = g. \tag{1}$$

Show that (1) admits a unique solution $u^* \in H^1(\Omega)$. Define the operator $A : H^{\frac{1}{2},2}(\partial\Omega) \to H^1(\Omega)$, $g \mapsto Ag := u^*$. Show that A is a bounded linear operator.

Exercise 18

Let the double sequence $(a_{ij})_{i,j\in\mathbb{N}}$ be such that $\sum_{i,j=1}^\infty |a_{ij}|^2 < \infty$. Assume that the matrix $A_N := (a_{ij})_{1\leq i,j\leq N}$ is positive definite for all $N \in \mathbb{N}$. Show that for every $f = (f_i)_{i\in\mathbb{N}} \in l^2$ the equation

$$u_i + \sum_{j=1}^{\infty} a_{ij} u_j = f_i \qquad (i \in \mathbb{N})$$

has a unique solution $u = (u_i)_{i\in\mathbb{N}} \in l^2$.

Hint: Here $l^2 = \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{R}, \sum_{i=1}^\infty |x_i|^2 < \infty\}$ and $\langle x, y\rangle_{l^2} = \sum_{i=1}^\infty x_i y_i$. You can use without proof that $(l^2, ||\cdot||_{l^2})$ is a Hilbert space.