Karlsruhe Institute of Technology (KIT) SS 2020
Institute for Analysis 10.06.2020
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Boundary and Eigenvalue Problems

Exercise sheet 7

Exercise 19

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Determine the weak formulation of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^2(\Omega)$. Show that a weak solution exists if and only if $\int_\Omega f dx = 0$.

Exercise 20

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $a_{ij}, b_i, c \in L^\infty(\Omega)$, $(i, j = 1, \ldots, n)$ and $f \in L^2(\Omega)$. Determine the weak formulation of the boundary value problem

$$\begin{cases} \Delta^2 u - \operatorname{div}(A\nabla u) + b \cdot \nabla u + cu = f & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

Derive an alternative theorem for this problem.

Hint: First prove a suitable Gårding inequality.

Exercise 21

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $S, T : H \to H$ be linear operators. Prove or provide a counterexample for each of the following statements:

1. If S is bounded and T is compact, then $S \circ T$ and $T \circ S$ are compact.
2. If T is compact, then T^2 is compact.
3. If T^2 is compact, then T is compact.

   Hint: Consider the Hilbert space l^2 and the operator $T(x_1, x_2, \ldots) = (x_2, 0, x_4, 0, x_6, \ldots)$.
4. If $\dim(T(H)) < \infty$, then T is compact.

   Hint: Let H be an infinite dimensional Hilbert space with a Hamel basis $B = \{x_i : i \in I\}$ and $\mathbb{N} \subset I$. Consider the linear operator with the property $Tx_i = ix_1$, $i \in \mathbb{N}$.
5. If $\dim(T(H)) < \infty$ and T is bounded, then T is compact.