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Boundary and Eigenvalue Problems

Exercise sheet 8

Exercise 22

Let $u \in H_0^1(0,1)$ be a weak solution of the boundary value problem

$$-u'' + b(x)u' + c(x)u = f(x) \quad \text{in } (0,1), \quad u(0) = u(1) = 0,$$

where $b, c \in L^\infty(0,1)$ and $f \in L^2(0,1)$. Show that $u \in H^2(0,1)$ and u satisfies the equation $-u''(x) + b(x)u'(x) + c(x)u(x) = f(x)$ pointwise almost everywhere in $(0,1)$.

Hint: Show that if $u \in H_0^1(0,1)$ and

$$\left| \int_0^1 u'\varphi' \, dx \right| \leq C\|\varphi\|_{L^2} \quad (\varphi \in H_0^1(0,1))$$

for a constant $C > 0$, then $u \in H^2(0,1)$.

Exercise 23

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Consider the following eigenvalue problems for $u \in C^2(\overline{\Omega})$:

$$\begin{cases} -\Delta u - \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases} \tag{1}$$

$$\begin{cases} -\Delta u - \lambda u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

1. Show that there exists no $u \in C^2(\overline{\Omega}) \setminus \{0\}$ which is an eigenfunction of both problems (1) and (2).

   Hint: Let u be a solution of the problem (1). Show that

   - $\int_\Omega |\nabla u|^2 \, dx = \lambda \int_\Omega |u|^2 \, dx$.
   - $\lambda \int_\Omega |u|^2 \, dx = \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 (x \cdot \nu) \, d\sigma$. *Hint:* Integrate $(\Delta u + \lambda u)(\nabla u \cdot x)$ and apply integration by parts. Sine u is a solution of (1), use without proof $\nabla u = \left(\frac{\partial u}{\partial \nu}\right) \nu$ on $\partial\Omega$.

2. Let $\Omega = (0,1)^2$. Calculate all eigenvalues and eigenfunctions of the problems (1) and (2).

Exercise 24

In the following let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Proof the following regularity theorem:

Theorem 1 (Interior regularity)**.** *Let $Lu = -\Delta u + cu$. Assume that $c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. Suppose that $u \in H^1(\Omega)$ is a weak solution of the elliptic PDE*

$$Lu = f \quad \textit{in } \Omega,$$

i.e.

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega c u \varphi \, dx = \int_\Omega f \varphi \, dx, \quad \textit{for all} \;\; \varphi \in H_0^1(\Omega).$$

Then $u \in H^2_{loc}(\Omega)$ *and for each open subset* $V \subset\subset \Omega$ *we have the estimate*

$$\|u\|_{H^2(V)} \leqslant C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right)$$

with a constant C *depending only on* V*,* Ω *and* c*.*

Hint: Use the following definition and theorem without proof.

Definition. Let $u \in L^1_{loc}(\Omega)$ and $V \subset\subset \Omega$.

(i) The i-th difference quotient of size h is

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} \quad (i = 1, \ldots, n)$$

for $x \in V$ and $0 < |h| < \operatorname{dist}(V, \partial\Omega)$.

(ii) $D^h u := \left(D_1^h u, \ldots, D_n^h u \right)$.

Theorem 2. *(i) Suppose* $u \in H^1(\Omega)$*. Then for each* $V \subset\subset \Omega$*,*

$$\|D^h u\|_{L^2(V)} \leqslant C \|\nabla u\|_{L^2(\Omega)}$$

for some constant $C > 0$ *and all* $0 < |h| < \operatorname{dist}(V, \partial\Omega)$*.*

(ii) Assume $u \in L^2(V)$ *and that there exists a constant* $C > 0$ *such that*

$$\|D^h u\|_{L^p(V)} \leqslant C$$

for all $0 < |h| < \frac{1}{2} \operatorname{dist}(V, \partial\Omega)$*. Then*

$$u \in H^1(V) \quad \textit{and} \quad \|\nabla u\|_{L^p(V)} \leqslant C.$$

Prove Theorem 1 by the following steps:

1. Let $V \subset\subset \Omega$. There exists an open set W such that $V \subset\subset W \subset\subset \Omega$. Let $\zeta : \mathbb{R}^n \to \mathbb{R}$ be a smooth cutoff function satisfying $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ in V and $\zeta \equiv 0$ in $\mathbb{R}^n \backslash W$.
2. Let $|h| > 0$ be sufficiently small and let $k \in \{1, \ldots, n\}$. Insert $v = -D_k^{-h}(\zeta^2 D_k^h u)$ into the weak formulation of $Lu = f$ to derive $A = B$, where
$$A := \int_\Omega \nabla u \cdot \nabla v \, dx \quad \text{and} \quad B := \int_\Omega (f - cu) v \, dx.$$
3. Show that $A \geqslant \frac{1}{2} \int_\Omega \zeta^2 \left| D_k^h D u \right|^2 dx - C \int_\Omega |\nabla u|^2 \, dx$ and $|B| \leqslant \frac{1}{4} \int_\Omega \zeta^2 |D_k^h \nabla u|^2 \, dx + C \int_\Omega f^2 + u^2 \, dx$.
4. Conclude that $u \in H^2(V)$ and $\|u\|_{H^2(V)} \leqslant C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)})$.
5. Consider a cutoff function ζ_2 such that $\zeta_2 \equiv 1$ in W, $0 \leqslant \zeta_2 \leqslant 1$ and $\operatorname{supp}(\zeta_2) \subset \Omega$. Insert $v_2 = \zeta_2^2 u$ into the weak formulation of $Lu = f$ to derive the inequality
$$\|u\|_{H^1(W)} \leqslant C \left(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \right).$$