Boundary and Eigenvalue Problems
Exercise sheet 9

Exercise 25

Let $X, Y$ be Banach spaces and denote by $\mathcal{B}(X,Y)$ and $\mathcal{K}(X,Y)$ the spaces of all bounded linear and all compact linear operators from $X$ to $Y$, each of them endowed with the operator norm $\|A\| := \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$ ($A \in \mathcal{B}(X,Y)$, resp. $A \in \mathcal{K}(X,Y)$). Show that

(i) $\mathcal{B}(X,Y)$ is a Banach space.

(ii) $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$.

Exercise 26

Let $k \in L^2((0,1) \times (0,1))$. Define

$$(Ku)(x) := \int_0^1 k(x,y)u(y) \, dy \quad (u \in L^2(0,1), \ x \in (0,1)).$$

Show that $K : L^2(0,1) \to L^2(0,1)$ is compact.

Hint: Approximate $k(x,y)$ by step functions $k_N(x,y)$ in $L^2((0,1) \times (0,1))$. Show that the operators

$$(K_Nu)(x) := \int_0^1 k_N(x,y)u(y) \, dy \quad (u \in L^2(0,1), \ x \in (0,1)).$$

are compact and use Exercise 25.

Exercise 27

(i) Let $H$ be an infinite dimensional inner product space. Let $A : D(A) \to H$ be a linear and symmetric operator. Suppose that $D(A)$ is dense in $H$ and assume that $H$ admits an orthonormal basis $(u_n)_{n \in \mathbb{N}}$ of eigenfunctions $u_n \in D(A)$ of $A$ and that the sequence of the corresponding eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ converges to $+\infty$ with $\lambda_1 \leq \lambda_2 \leq \ldots$. Show that for all $n \in \mathbb{N}$ the following holds:

$$\lambda_n = \min_{U \subset D(A)} \max_{u \in U, u \neq 0} \frac{\langle Au, u \rangle}{\langle u, u \rangle} \quad \text{(Poincaré’s min-max principle).}$$

(ii) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We define operators $Au = -\Delta u$ and $Bu = -\Delta u + cu$ with the real-valued function $c \in L^\infty(\Omega)$ and $D(A) = D(B) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega)$. The eigenvalue sequences $(\lambda_i)$ of $A$ and $(\mu_i)$ of $B$, which both converge to infinity, can be ordered by magnitude: $\lambda_1 \leq \lambda_2 \leq \ldots$ and $\mu_1 \leq \mu_2 \leq \ldots$.

Show that $\lambda_i + c \leq \mu_i \leq \lambda_i + \overline{c}$, for all $i \in \mathbb{N}$, where $\underline{c}, \overline{c} \in \mathbb{R}$ are such that $\underline{c} \leq c \leq \overline{c}$ almost everywhere in $\Omega$. 

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