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Boundary and Eigenvalue Problems

Exercise sheet 9

Exercise 25

Let X, Y be Banach spaces and denote by $\mathcal{B}(X,Y)$ and $\mathcal{K}(X,Y)$ the spaces of all bounded linear and all compact linear operators from X to Y, each of them endowed with the operator norm $\|A\| := \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$ ($A \in \mathcal{B}(X,Y)$, resp. $A \in \mathcal{K}(X,Y)$). Show that

(i) $\mathcal{B}(X,Y)$ is a Banach space.

(ii) $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$.

Exercise 26

Let $k \in L^2((0,1)\times(0,1))$. Define

$$(Ku)(x) := \int_0^1 k(x,y)u(y)\,dy \qquad (u \in L^2(0,1),\ x \in (0,1)).$$

Show that $K : L^2(0,1) \longrightarrow L^2(0,1)$ is compact.

Hint: Approximate $k(x,y)$ by step functions $k_N(x,y)$ in $L^2((0,1)\times(0,1))$. Show that the operators

$$(K_N u)(x) := \int_0^1 k_N(x,y)u(y)\,dy \qquad (u \in L^2(0,1),\ x \in (0,1)).$$

are compact and use Exercise 25.

Exercise 27

(i) Let H be an infinite dimensional inner product space. Let $A : D(A) \to H$ be a linear and symmetric operator. Suppose that $D(A)$ is dense in H and assume that H admits an orthonormal basis $(u_n)_{n\in\mathbb{N}}$ of eigenfunctions $u_n \in D(A)$ of A and that the sequence of the corresponding eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ converges to $+\infty$ with $\lambda_1 \leq \lambda_2 \leq ....$ Show that for all $n \in \mathbb{N}$ the following holds:

$$\lambda_n = \min_{\substack{U\subset D(A) \\ \text{subspace,} \\ \dim(U)=n}} \max_{u\in U, u\neq 0} \frac{\langle Au, u\rangle}{\langle u, u\rangle} \qquad \textit{(Poincaré's min-max principle).}$$

(ii) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. We define operators $Au = -\Delta u$ and $Bu = -\Delta u + cu$ with the real-valued function $c \in L^\infty(\Omega)$ and $D(A) = D(B) = H^2(\Omega)\cap H_0^1(\Omega) \subset L^2(\Omega)$. The eigenvalue sequences (λ_i) of A and (μ_i) of B, which both converge to infinity, can be ordered by magnitude: $\lambda_1 \leq \lambda_2 \leq \ldots$ and $\mu_1 \leq \mu_2 \leq \ldots$.

Show that $\lambda_i + \underline{c} \leq \mu_i \leq \lambda_i + \overline{c}$, for all $i \in \mathbb{N}$, where $\underline{c}$, $\overline{c} \in \mathbb{R}$ are such that $\underline{c} \leq c \leq \overline{c}$ almost everywhere in Ω.