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Institute for Analysis
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Bifurcation Theory

Problem Sheet 1

Problem 1 (Bifurcation Diagrams)

Draw bifurcation diagrams for the following equations:

(a) $x^3 + 2\lambda x^2 + \lambda^3 x = 0$ $(\lambda, x \in \mathbb{R})$,

(b) $x + \sinh(\lambda x) = 0$ $(\lambda, x \in \mathbb{R})$.

Problem 2 (Bifurcation with respect to different topologies)

Consider the function $u_1 : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{\sqrt{2}}{\cosh(x)}$.

(a) Prove that $u_1 \in W^{2,q}(\mathbb{R})$ for all $q \in [1, \infty]$.

(b) Prove that u_1 solves the ODE $-u'' + u - u^3 = 0$ on $\mathbb{R}$.

(c) Find a nontrivial family of solutions $\mathcal{T} := \{(u_\lambda, \lambda) :\ \lambda > 0\}$ of

$$\begin{cases} -u'' + \lambda u - u^3 = 0, \\ u \in W^{2,q}(\mathbb{R}) \end{cases} \tag{1}$$

and, for each $q \in [1, \infty]$, decide whether it bifurcates from the trivial branch at $(0, 0)$ with respect to $\|\cdot\|_{W^{2,q}(\mathbb{R})}$.

Problem 3 (Density of $C_0^\infty(\mathbb{R}^n)$)

For $1 \le p < \infty$, $f \in L^p(\mathbb{R}^n), g \in L^1(\mathbb{R}^n)$, we define the *convolution* $f * g \in L^p(\mathbb{R}^n)$

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x - y) g(y)\, \mathrm{d}y.$$

Then, *Young's inequality* states that $\|f * g\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}$.

(a) Let $1 \le p < \infty$, $f \in L^p(\mathbb{R}^n)$ and for $y \in \mathbb{R}^n$, define the translation $(\tau_y f)(x) := f(x-y)$, $x \in \mathbb{R}^n$. Argue that $\tau_y f \in L^p(\mathbb{R}^n)$ and prove

$$\|f - \tau_y f\|_{L^p(\mathbb{R}^n)} \to 0 \qquad \text{as } y \to 0.$$

(b) Let $1 \le p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $\psi_\varepsilon(x) := \varepsilon^{-n}\psi\left(\frac{1}{\varepsilon}x\right)$ for $\varepsilon > 0$ and some $\psi \in C_0^\infty(\mathbb{R}^n)$. Prove that

$$\|\psi_\varepsilon * f - f\|_{L^p(\mathbb{R}^n)} \to 0 \quad (\varepsilon \searrow 0).$$

(c) Conclude that $C_0^\infty(\mathbb{R}^n)$ is dense both in $L^p(\mathbb{R}^n)$ and in $W^{k,p}(\mathbb{R}^n)$ for $1 \le p < \infty$, $k \in \mathbb{N}$.

(d) Argue that $C_0^\infty(\mathbb{R}^n)$ is dense neither in $L^\infty(\mathbb{R}^n)$ nor in $W^{k,\infty}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$.

Some Hints:
For parts (a) and (b), recall that $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. (Proof in the Problem Class.) In part (c), you can use without proof that $f \in L^p(\mathbb{R}^n), g \in C_0^\infty(\mathbb{R}^n)$ implies that $f * g \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and that, if additionally f has compact support, $f * g \in C_0^\infty(\mathbb{R}^n)$.

Some notes on organisation:

Problem Sheets

- New problem sheets will be published on the webpage on Wednesdays.
  `http://www.math.kit.edu/iana2/edu/bifurcation2017s/`
- Problem sheets can be handed in for grading at the beginning of the problem classes. Grading is not compulsory.

For questions, comments, ...

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