Bifurcation Theory

Problem Sheet 1

Problem 1 (Bifurcation Diagrams)
Draw bifurcation diagrams for the following equations:
(a) \( x^3 + 2\lambda x^2 + \lambda^3 x = 0 \) (\( \lambda, x \in \mathbb{R} \)),
(b) \( x + \sinh(\lambda x) = 0 \) (\( \lambda, x \in \mathbb{R} \)).

Problem 2 (Bifurcation with respect to different topologies)
Consider the function \( u_1 : \mathbb{R} \to \mathbb{R}, \ x \mapsto \frac{\sqrt{2}}{\cosh(x)} \).
(a) Prove that \( u_1 \in W^{2,q}(\mathbb{R}) \) for all \( q \in [1,\infty] \).
(b) Prove that \( u_1 \) solves the ODE \(-u'' + u - u^3 = 0\) on \( \mathbb{R} \).
(c) Find a nontrivial family of solutions \( \mathcal{T} := \{(u_\lambda, \lambda) : \lambda > 0\} \) of
\[
\begin{align*}
-u'' + \lambda u - u^3 &= 0, \quad (1) \\
u &\in W^{2,q}(\mathbb{R})
\end{align*}
\]
and, for each \( q \in [1,\infty] \), decide whether it bifurcates from the trivial branch at \((0,0)\) with respect to \( \| \cdot \|_{W^{2,q}(\mathbb{R})} \).

Problem 3 (Density of \( C^\infty_0(\mathbb{R}^n) \))
For \( 1 \leq p < \infty \), \( f \in L^p(\mathbb{R}^n) \), \( g \in L^1(\mathbb{R}^n) \), we define the convolution \( f * g \in L^p(\mathbb{R}^n) \)
\[
(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y) \, dy.
\]
Then, Young’s inequality states that \( \| f * g \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^p(\mathbb{R}^n)} \| g \|_{L^1(\mathbb{R}^n)} \).
(a) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$ and for $y \in \mathbb{R}^n$, define the translation $(\tau_y f)(x) := f(x-y)$, $x \in \mathbb{R}^n$. Argue that $\tau_y f \in L^p(\mathbb{R}^n)$ and prove

$$\|f - \tau_y f\|_{L^p(\mathbb{R}^n)} \to 0 \quad \text{as} \quad y \to 0.$$ 

(b) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $\psi_\varepsilon(x) := \varepsilon^{-n} \psi\left(\frac{1}{\varepsilon} x\right)$ for $\varepsilon > 0$ and some $\psi \in C^\infty_0(\mathbb{R}^n)$. Prove that

$$\|\psi_\varepsilon \ast f - f\|_{L^p(\mathbb{R}^n)} \to 0 \quad (\varepsilon \downarrow 0).$$

(c) Conclude that $C^\infty_0(\mathbb{R}^n)$ is dense both in $L^p(\mathbb{R}^n)$ and in $W^{k,p}(\mathbb{R}^n)$ for $1 \leq p < \infty$, $k \in \mathbb{N}$.

(d) Argue that $C^\infty_0(\mathbb{R}^n)$ is dense neither in $L^\infty(\mathbb{R}^n)$ nor in $W^{k,\infty}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$.

Some Hints:
For parts (a) and (b), recall that $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. (Proof in the Problem Class.) In part (c), you can use without proof that $f \in L^p(\mathbb{R}^n)$, $g \in C^\infty_0(\mathbb{R}^n)$ implies that $f \ast g \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and that, if additionally $f$ has compact support, $f \ast g \in C^\infty_0(\mathbb{R}^n)$.

Some notes on organisation:

Problem Sheets

- New problem sheets will be published on the webpage on Wednesdays. http://www.math.kit.edu/iana2/edu/bifurcation2017s/

- Problem sheets can be handed in for grading at the beginning of the problem classes. Grading is not compulsory.

For questions, comments, ...

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