Problem 1 (Bifurcation Diagrams)

Draw bifurcation diagrams for the following equations:

(a) \(x^3 + 2\lambda x^2 + \lambda^3 x = 0\ (\lambda, x \in \mathbb{R})\),

(b) \(x + \sinh(\lambda x) = 0\ (\lambda, x \in \mathbb{R})\).

Solution

Let us first note that both equations are satisfied by the trivial solution family \(\{(0, \lambda) : \lambda \in \mathbb{R}\}\). We now aim to find (nontrivial) solutions \((x, \lambda) \in \mathbb{R} \times \mathbb{R}, x \neq 0\).

(a) We define the auxiliary function

\[ f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f(x, \lambda) := x^2 + 2\lambda x + \lambda^3 \]

and calculate its zeros explicitly:

\[ 0 = f(x, \lambda) \iff 0 = (x + \lambda)^2 + \lambda^3 - \lambda^2 \iff \lambda \leq 1, \quad x \in \left\{-\lambda \pm \sqrt{\lambda^2 - \lambda^3}\right\}. \]

This yields the following bifurcation diagram which shows that \((0, 0)\) is a bifurcation point:

![Bifurcation Diagram](image-url)
(b) For every $\lambda \in \mathbb{R}$, we introduce the smooth function

$$ g_\lambda : \mathbb{R} \to \mathbb{R}, \quad g_\lambda(x) := x + \sinh(\lambda x). $$

We already know that $g_\lambda(0) = 0$, which corresponds to the trivial branch. Further,

$$ g'_\lambda(x) = 1 + \lambda \cosh(\lambda x) \quad (x \in \mathbb{R}) $$

implies (due to $\cosh(y) > 1, y \in \mathbb{R} \setminus \{0\}$) that $g_\lambda$ is strictly monotone for $\lambda \geq 0$ and $\lambda \leq -1$, so in these cases, 0 is the only zero of $g_\lambda$. In the remaining cases, we prove:

(1) **Assertion:** For $-1 < \lambda < 0$, there exists $x_\lambda > 0$ with the property that

$$ g_\lambda^{-1}(\{0\}) = \{-x_\lambda, 0, x_\lambda\}. $$

**Proof:** Let $-1 < \lambda < 0$. As the function $g_\lambda$ is odd, we only have to prove that it has exactly one positive zero $x_\lambda$.

**Existence:** As $\lambda < 0$, we conclude $\lim_{x \to \infty} g_\lambda(x) = -\infty$. Further, $\lambda > -1$ implies that $g'_\lambda(0) = 1 + \lambda > 0$, and hence, there exists $\delta_\lambda > 0$ with $g_\lambda(\delta_\lambda) = \max_{x > 0} g_\lambda(x) > 0$. Since $g_\lambda$ is continuous, Bolzano’s theorem now yields the existence of a zero $x_\lambda > \delta_\lambda$.

**Uniqueness:** This is a consequence of the fact that $g_\lambda$ is strictly concave on $(0, \infty)$ due to

$$ g''_\lambda(x) = \lambda^2 \sinh(\lambda x) < 0 \quad (x > 0). $$

(Hence, the first derivative is strictly decreasing, and $g_\lambda$ has at most one critical point in $(0, \infty)$, which is $\delta_\lambda \in (0, x_\lambda)$. We infer that $g_\lambda$ is strictly monotone on both $(0, \delta_\lambda)$ and $(\delta_\lambda, \infty)$, which proves the assertion.)

(2) **Assertion:** The mapping $\lambda \mapsto x_\lambda \ (-1 < \lambda < 0)$ is continuous.

**Proof:** This is a consequence of the Implicit Function Theorem, which can be applied since

$$ g_\lambda(x_\lambda) = 0, \quad g'_\lambda(x_\lambda) < 0 $$

(where the latter inequality results from strict concavity, see above).

(3) **Assertion:** We have the following one-sided limits:

$$ \lim_{\lambda \to -1^+} x_\lambda = 0, \quad \lim_{\lambda \to 0^-} x_\lambda = \infty. $$

**Proof:** For $\lambda \in (-1, 0)$, we estimate by power series expansion:

$$ x_\lambda = -\sinh(\lambda x_\lambda) = \sinh(|\lambda|x_\lambda) \geq |\lambda|x_\lambda + \frac{1}{6} |\lambda|^3 x_\lambda^3, $$
which yields
\[ x^2_\lambda \leq \frac{6}{|\lambda|^3} (1 - |\lambda|) \to 0 \quad (\lambda \to -1, \quad \lambda > -1). \]

On the other hand, we recall that \( g'(x_\lambda) < 0 \) and obtain by inserting \( g_\lambda(x_\lambda) = 0 \)
\[
0 > 1 - |\lambda| \cosh(\lambda x_\lambda) = 1 - |\lambda| \sqrt{1 + \sinh^2(\lambda x_\lambda)} = 1 - |\lambda| \sqrt{1 + x^2_\lambda}.
\]

This can only hold in the limit \( \lambda \to 0, \lambda < 0 \) if, at the same time, \( x_\lambda \to \infty \). ■

This yields the following bifurcation diagram:
Problem 2 (Bifurcation with respect to different topologies)

Consider the function \( u_1 : \mathbb{R} \to \mathbb{R}, \ x \mapsto \sqrt{2} \cosh(x) \).

(a) Prove that \( u_1 \in W^{2,q}(\mathbb{R}) \) for all \( q \in [1, \infty] \).

(b) Prove that \( u_1 \) solves the ODE \(-u'' + u - u^3 = 0\) on \( \mathbb{R} \).

(c) Find a nontrivial family of solutions \( \mathcal{T} := \{(u_\lambda, \lambda) : \lambda > 0\} \) of

\[
\begin{aligned}
-\frac{\dd}{\dd x} + u - u^3 &= 0, \\
\end{aligned}
\]

and, for each \( q \in [1, \infty] \), decide whether it bifurcates from the trivial branch at \((0,0)\) with respect to \( \|\cdot\|_{W^{2,q}(\mathbb{R})} \).

Solution

(a) Clearly, \( u_1 \) is a smooth function. For \( x \in \mathbb{R} \), we have (with \( \sinh^2(x) + 1 = \cosh^2(x) \))

\[
\begin{aligned}
u_1'(x) &= -\frac{\sqrt{2}}{\cosh^2(x)} \cdot \sinh(x), \\
u_1''(x) &= \frac{2\sqrt{2}}{\cosh^3(x)} \cdot \sinh^2(x) = \frac{\sqrt{2}}{\cosh^2(x)} \cdot [2 \sinh^2(x) - \cosh^2(x)] \\
&= \frac{\sqrt{2}}{\cosh^2(x)} \cdot [\cosh^2(x) - 2] = u_1(x) - u_1(x)^3. \\
\end{aligned}
\]

(\( \Diamond \))

As \( u_1 \in C^\infty(\mathbb{R}) \), weak derivatives of second (in fact, of every) order exist and agree with the classical ones computed above.

Let \( q \in [1, \infty] \). It remains to check integrability, i.e. whether \( u_1, u_1', u_1'' \in L^q(\mathbb{R}) \). We exploit that, for \( x \in \mathbb{R} \),

\[
\cosh(x) = \frac{e^x + e^{-x}}{2} \geq \frac{1}{2} e^{\|x\|} \quad \text{and} \quad |\sinh(x)| = \frac{|e^x - e^{-x}|}{2} \leq \frac{e^x + e^{-x}}{2} = \cosh(x),
\]

and estimate as follows:

\[
\begin{aligned}
|u_1(x)| &= \frac{\sqrt{2}}{\cosh(x)} \leq 2\sqrt{2} e^{-\|x\|}, \\
|u_1'(x)| &= \frac{\sqrt{2}}{\cosh^2(x)} \cdot |\sinh(x)| \leq \frac{\sqrt{2}}{\cosh(x)} \leq 2\sqrt{2} e^{-\|x\|}, \\
|u_1''(x)| &= \frac{\sqrt{2}}{\cosh^3(x)} \cdot |\cosh^2(x) - 2| \leq \frac{\sqrt{2}}{\cosh(x)} + \left( \frac{\sqrt{2}}{\cosh(x)} \right)^3 \leq 2\sqrt{2} e^{-\|x\|} + 16\sqrt{2} e^{-3\|x\|} \\
&\leq 18\sqrt{2} e^{-\|x\|}.
\end{aligned}
\]

Since \( e^{-\|x\|} \in L^q(\mathbb{R}) \), we conclude that \( u_1 \in W^{2,q}(\mathbb{R}) \) as claimed.
(b) This has been shown in the course of (a), equation (\(\diamond\)).

(c) **First Step:** We construct a branch of nontrivial solutions \((u_\lambda, \lambda) (\lambda > 0)\) by scaling.

For \(\lambda > 0\) and \(x \in \mathbb{R}\), we define

\[
u_\lambda(x) := \sqrt{\lambda} \cdot u_1(\sqrt{\lambda} x) = \frac{\sqrt{2\lambda}}{\cosh(\sqrt{\lambda} x)} \quad (x \in \mathbb{R}, \lambda > 0).
\]

As in the first step, we see that \(u_\lambda \in W^{2,q}(\mathbb{R})\) for every \(q \in [1, \infty)\) and note that \(u_\lambda \neq 0\). Moreover, \(u_\lambda\) is a smooth function and we have for \(x \in \mathbb{R}\)

\[-u''_\lambda(x) + \lambda u_\lambda(x) - u^3_\lambda(x) = \lambda^{\frac{3}{2}} \left( -u''_1(\sqrt{\lambda} x) + u_1(\sqrt{\lambda} x) - u_1(\sqrt{\lambda} x)^3 \right) = 0.\]

Hence, for all \(\lambda > 0\), \(u_\lambda \in W^{2,q}(\mathbb{R}) \cap C^\infty(\mathbb{R})\) is a classical solution to

\[-u'' + \lambda u = u^3 \quad \text{in } \mathbb{R}.
\]

So we have found a family \(\mathcal{T} := \{(u_\lambda, \lambda) : \lambda > 0\}\) of nontrivial solutions to (1).

**Second Step:** We discuss whether \(\mathcal{T}\) bifurcates from \((0,0)\).

To this end, we fix \(q \in [1, \infty]\), calculate the norms \(\|u^{(j)}_\lambda\|_{L^q(\mathbb{R})} \cdot j = 0, 1, 2\), and discuss the limit \(\lambda \to 0\). We have for \(\lambda > 0\) and \(x \in \mathbb{R}\)

\[u_\lambda(x) = \sqrt{\lambda} \cdot u_1(\sqrt{\lambda} x), \quad u'_\lambda(x) = \lambda \cdot u'_1(\sqrt{\lambda} x), \quad u''_\lambda(x) = \lambda \sqrt{\lambda} \cdot u''_1(\sqrt{\lambda} x).
\]

Hence, for \(j = 0, 1, 2\),

\[\|u^{(j)}_\lambda\|_{L^\infty(\mathbb{R})} = \lambda^{\frac{j+2}{2}} \|u^{(j)}_1\|_{L^\infty(\mathbb{R})} \xrightarrow{\lambda \to 0} 0,
\]

and for \(1 \leq q < \infty\),

\[\|u^{(j)}_\lambda\|^q_{L^q(\mathbb{R})} = \int_\mathbb{R} |u^{(j)}_\lambda(x)|^q \, dx = \int_\mathbb{R} |\lambda^{\frac{j+1}{2}} \cdot u^{(j)}_1(\sqrt{\lambda} x)|^q \, dx \xrightarrow{\lambda \to 0} \lambda^{\frac{q+2-j}{2}} \int_\mathbb{R} |u^{(j)}_1(y)|^q \, dy
\]

\[= \lambda^{\frac{q-j}{2}} \|u^{(j)}_1\|^q_{L^q(\mathbb{R})} \xrightarrow{\lambda \to 0} \begin{cases} 0, & 1 < q < \infty \text{ or } j = 1, 2, \\ \|u_1\|^q_{L^q(\mathbb{R})} > 0, & q = 1 \text{ and } j = 0. \end{cases}
\]

We conclude that

\[\lim_{\lambda \to 0} \|u_\lambda\|_{W^{2,q}(\mathbb{R})} = 0 \iff 1 < q \leq \infty,
\]

and bifurcation from the trivial branch with respect to \(\|\cdot\|_{W^{2,q}(\mathbb{R})}\) occurs if and only if \(q \in (1, \infty]\).
Problem 3 (Density of $C^\infty_0(\mathbb{R}^n)$)

For $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, we define the convolution $f * g \in L^p(\mathbb{R}^n)$

$$ (f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y) \, dy. $$

Then, Young’s inequality states that $\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}$.

(a) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$ and for $y \in \mathbb{R}^n$, define the translation $(\tau_y f)(x) := f(x - y)$, $x \in \mathbb{R}^n$. Argue that $\tau_y f \in L^p(\mathbb{R}^n)$ and prove

$$ \|f - \tau_y f\|_{L^p(\mathbb{R}^n)} \to 0 \quad \text{as } y \to 0.$$

(b) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $\psi_\varepsilon(x) := \varepsilon^{-n} \psi\left(\frac{x}{\varepsilon}\right)$ for $\varepsilon > 0$ and some $\psi \in C^\infty_0(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x) \, dx = 1$ and $\psi \geq 0$. Prove that

$$ \|\psi_\varepsilon \ast f - f\|_{L^p(\mathbb{R}^n)} \to 0 \quad (\varepsilon \searrow 0).$$

(c) Conclude that $C^\infty_0(\mathbb{R}^n)$ is dense both in $L^p(\mathbb{R}^n)$ and in $W^{k,p}(\mathbb{R}^n)$ for $1 \leq p < \infty$, $k \in \mathbb{N}$.

(d) Argue that $C^\infty_0(\mathbb{R}^n)$ is dense neither in $L^\infty(\mathbb{R}^n)$ nor in $W^{k,\infty}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$.

Some Hints:

For parts (a) and (b), recall that $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. (Proof in the Problem Class.)

In part (c), you can use without proof that $f \in L^p(\mathbb{R}^n)$, $g \in C^\infty_0(\mathbb{R}^n)$ implies that $f \ast g \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and that, if additionally $f$ has compact support, $f \ast g \in C^\infty_0(\mathbb{R}^n)$.

Solution

(a) Let $y \in \mathbb{R}^n$. As a direct consequence of the invariance of the Lebesgue measure under translations, $\tau_y f$ is measurable and the coordinate transformation $\mathbb{R}^n \to \mathbb{R}^n$, $x - y \mapsto x$ yields $\|\tau_y f\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$.

First Step: We prove the convergence property for a compactly supported continuous function $g \in C_0(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$.

As $g$ is continuous with compact support, it is in particular bounded, and we choose $C_g > 0$ with $|g(x)| \leq C_g$ for all $x \in \mathbb{R}^n$. Moreover, we denote the (compact) support of $g$ by $K_g := \text{spt } g$ and introduce the Minkowski sum $K_g(q) := K_g + \overline{B}_q(0)$ for $q > 0$. We aim to prove

$$ \left(\int_{\mathbb{R}^n} |g(x - y) - g(x)|^p \, dx\right)^{\frac{1}{p}} \to 0 \quad \text{as } y \to 0 \quad \forall q > 0,$$
by dominated convergence. By continuity of $g$, we have pointwise convergence
\[ |g(x - y) - g(x)|^p \to 0 \]
as $y \to 0$ for every fixed $x \in \mathbb{R}^n$. Moreover, considering $y \in \mathbb{R}^n$ with $|y| \leq 1$ only, we have the estimate
\[ |g(x - y) - g(x)|^p \leq (2C_g)^p \cdot 1_{K_g(1)}(x) \]
where the right-hand side is integrable due to compactness of $K_g$. Thus, an application of dominated convergence proves the assertion.

**Second Step**: We now consider $f \in L^p(\mathbb{R}^n)$.

Let $\eta > 0$. By the density of compactly supported continuous functions in $L^p(\mathbb{R}^n)$, we find $g \in C_0(\mathbb{R}^n)$ with the property that $\|f - g\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{2}$. Moreover, the first step yields $\delta > 0$ with the property that $\|\tau_y g - g\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{3}$ for every $y \in \mathbb{R}^n, |y| < \delta$. The assertion now follows from the triangle inequality; for $y \in \mathbb{R}^n, |y| < \delta$ we have
\[
\|\tau_y f - f\|_{L^p(\mathbb{R}^n)} \leq \|\tau_y (f - g)\|_{L^p(\mathbb{R}^n)} + \|\tau_y g - g\|_{L^p(\mathbb{R}^n)} + \|g - f\|_{L^p(\mathbb{R}^n)}
\]
\[
= \|\tau_y g - g\|_{L^p(\mathbb{R}^n)} + 2 \|g - f\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{3} + 2 \cdot \frac{\eta}{3} = \eta.
\]
\[
\Box
\]

**b)** For $\varepsilon > 0$, we have
\[
\int_{\mathbb{R}^n} \psi_\varepsilon(x) \, dx = \int_{\mathbb{R}^n} \varepsilon^{-n} \psi \left( \frac{1}{\varepsilon} x \right) \, dx \xrightarrow{x = \varepsilon y} \int_{\mathbb{R}^n} \psi(y) \, dy = 1.
\]

We first use commutativity of the convolution $(\psi_\varepsilon * f = f * \psi_\varepsilon)$, then this result, afterwards Hölder’s inequality with $\frac{1}{p} + \frac{1}{p'} = 1$, finally Tonelli’s theorem and obtain
\[
\|\psi_\varepsilon * f - f\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} \left| \left( \int_{\mathbb{R}^n} \psi_\varepsilon(y) f(x - y) \, dy \right) - f(x) \right|^p \, dx
\]
\[
= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi_\varepsilon(y) (f(x - y) - f(x)) \, dy \right|^p \, dx
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi_\varepsilon(y)^{\frac{1}{p}} \cdot \psi_\varepsilon(y)^{\frac{1}{p'}} (f(x - y) - f(x)) \, dy \, dx
\]
\[
\leq \left( \int_{\mathbb{R}^n} \psi_\varepsilon(y)^{\frac{1}{p'}} \, dy \right)^{\frac{p}{p'}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi_\varepsilon(y)^{\frac{1}{p}} (f(x - y) - f(x)) \, dy \, dx
\]
\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \psi_\varepsilon(y) \left| f(x - y) - f(x) \right|^p \, dy \, dx
\]
\[
= \int_{\mathbb{R}^n} \psi_\varepsilon(y) \left( \int_{\mathbb{R}^n} \left| f(x - y) - f(x) \right|^p \, dx \right) \, dy
\]
\[
= \int_{\mathbb{R}^n} \psi_\varepsilon(y) \|\tau_y f - f\|_{L^p(\mathbb{R}^n)}^p \, dy
\]  
(2)
Now, let \( \eta > 0 \). By virtue of (a), we can choose \( \delta > 0 \) with the property that, for \( y \in \mathbb{R}^n \) with \( |y| \leq \delta \),
\[
\| \tau_\delta f - f \|_{L^p(\mathbb{R}^n)} < \eta. \tag{3}
\]
Moreover, by definition of \( \psi_\varepsilon \), we have that \( \psi(x) = 0 \iff \psi_\varepsilon(\varepsilon x) = 0 \) and thus
\[
\text{spt } \psi_\varepsilon = \{ \varepsilon x : x \in \text{spt } \psi \} = \varepsilon \cdot \text{spt } \psi.
\]
We therefore find \( \varepsilon_0 > 0 \) with the property that, for every \( \varepsilon \in (0, \varepsilon_0) \),
\[
\text{spt } \psi_\varepsilon \subseteq B_\delta(0). \tag{4}
\]
With that, we continue the estimate (2) and conclude for \( \varepsilon \in (0, \varepsilon_0) \)
\[
\| \psi_\varepsilon * f - f \|_{L^p(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} \psi_\varepsilon(y) \| \tau_\delta f - f \|_{L^p(\mathbb{R}^n)} \, dy \leq \int_{B_\delta(0)} \psi_\varepsilon(y) \| \tau_\delta f - f \|_{L^p(\mathbb{R}^n)} \, dy \leq \eta.
\]
This closes the proof of part (b).

(c) We let \( \psi, \psi_\varepsilon \in C_0^\infty(\mathbb{R}^n), \varepsilon > 0 \), as in part (b). (For the existence of such a function, we refer to the lecture *Analysis 2*.)

*First Step: \( C_0^\infty(\mathbb{R}^n) \) is dense in \( L^p(\mathbb{R}^n), 1 \leq p < \infty \).*

We let \( f \in L^p(\mathbb{R}^n) \) and \( \eta > 0 \).

As in the proof of part (a), we can choose \( g \in C_0(\mathbb{R}^n) \) with \( \| f - g \|_{L^p(\mathbb{R}^n)} < \frac{\eta}{2} \), and it suffices to find \( h \in C_0^\infty(\mathbb{R}^n) \) with \( \| g - h \|_{L^p(\mathbb{R}^n)} < \frac{\eta}{2} \). The hint implies that the convolutions satisfy \( \psi_\varepsilon * g \in C_0^\infty(\mathbb{R}^n) \), and part (b) gives
\[
\| \psi_\varepsilon * f - g \|_{L^p(\mathbb{R}^n)} \to 0 \quad (\varepsilon \searrow 0).
\]
Hence, choosing \( h := \psi_\varepsilon * g \) with \( \varepsilon > 0 \) so small that \( \| \psi_\varepsilon * g - g \|_{L^p(\mathbb{R}^n)} < \frac{\eta}{2} \), we have found a suitable function \( h \in C_0^\infty(\mathbb{R}^n) \).

*Second Step: \( C_0^\infty(\mathbb{R}^n) \) is dense in \( W^{k,p}(\mathbb{R}^n), 1 \leq p < \infty \) and \( k \in \mathbb{N} \).*

We introduce the space \( W^{k,p}_{\text{cpt}}(\mathbb{R}^n) := \{ u \in W^{k,p}(\mathbb{R}^n) : \text{spt } (u) \text{ is compact} \} \).

*Second step, first part: \( W^{k,p}_{\text{cpt}}(\mathbb{R}^n) \) is dense in \( W^{k,p}(\mathbb{R}^n) \).*

We consider \( u \in W^{k,p}(\mathbb{R}^n) \) and introduce a smooth cutoff function \( \chi \in C_0^\infty(\mathbb{R}^n) \) with \( 1_{B_1(0)} \leq \chi \leq 1_{B_2(0)} \). We define \( C := \max_{|\alpha| \leq k} \| \partial^\alpha \chi \|_\infty \) and, for \( R > 0 \) and \( x \in \mathbb{R}^n \),
\[
\chi_R(x) := \chi \left( \frac{x}{R} \right).
\]

\footnote{This does not take into account that \( W^{k,p}(\mathbb{R}^n) \) actually contains equivalence classes of functions. Strictly speaking, we demand that \( u \) has a representative whose support is compact, and henceforth consider that representative.}
Let $\eta > 0$. By monotone convergence, we find $R > 0$ with the property that, for every multiindex with $|\alpha| \leq k$,

$$
\|\partial^\alpha u\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \leq \frac{\eta}{2 \cdot \max\{C, 1\}}
$$

and, additionally, $R \geq 1$. We note that $u \cdot \chi_R \in W^{k,p}_{\text{cpt}}(\mathbb{R}^n)$ and estimate for an arbitrary multiindex with $|\alpha| \leq k$

$$
\|\partial^\alpha (u \cdot \chi_R) - \partial^\alpha u\|_{L^p(\mathbb{R}^n)} \leq \| (|\chi_R - 1|) \cdot \partial^\alpha u\|_{L^p(\mathbb{R}^n)} + \| u \cdot \partial^\alpha \chi_R\|_{L^p(\mathbb{R}^n \setminus B_R(0))} + \| u \cdot \partial^\alpha \chi_R\|_{L^p(B_{2R}(0) \setminus B_R(0))}
$$

$$
\leq \| \partial^\alpha u\|_{L^p(\mathbb{R}^n \setminus B_R(0))} + \frac{C}{R^{\alpha}} \cdot \| u\|_{L^p(\mathbb{R}^n \setminus B_R(0))}
$$

$$
\leq \frac{\eta}{2} + C \cdot \frac{\eta}{2C} = \eta
$$

where we have used that $1_{B_R(0)} \leq \chi_R \leq 1_{B_{2R}(0)}$ and $\partial^\alpha \chi_R(x) = R^{-|\alpha|}(\partial^\alpha \chi) \left( \frac{x}{R} \right)$. This proves the first part.

**Second step, second part:** $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}_{\text{cpt}}(\mathbb{R}^n)$ (in the topology of $W^{k,p}(\mathbb{R}^n)$).

We let $u \in W^{k,p}_{\text{cpt}}(\mathbb{R}^n)$ and $\eta > 0$. Taking $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ as in part (a), the hint now gives\footnote{Note that, if only $u \in W^{k,p}(\mathbb{R}^n)$, we would find $\psi_\varepsilon \ast u \in C^\infty(\mathbb{R}^n)$ - compact support is lost in the process of smoothing via convolution.} that $\psi_\varepsilon \ast u \in C_0^\infty(\mathbb{R}^n)$. For every multiindex with $|\alpha| \leq k$, part (b) implies

$$
\|\partial^\alpha (\psi_\varepsilon \ast u) - \partial^\alpha u\|_{L^p(\mathbb{R}^n)} = \|\psi_\varepsilon \ast \partial^\alpha u - \partial^\alpha u\|_{L^p(\mathbb{R}^n)} \to 0 \quad (\varepsilon \searrow 0).
$$

Thus, taking $\varepsilon > 0$ sufficiently small, we find that

$$
\|\psi_\varepsilon \ast u - u\|_{W^{k,p}(\mathbb{R}^n)} < \eta,
$$

which proves the assertion. Combining the first and the second part of step 2, the proof is complete. \qed
(d) **First Step:** $C_0^\infty(\mathbb{R}^n)$ is not dense in $L^\infty(\mathbb{R}^n)$.

We prove more generally that $C_0(\mathbb{R}^n)$ is not dense in $L^\infty(\mathbb{R}^n)$. We consider the step function $f := \mathbb{1}_{B_1(0)}$, which is an element of $L^\infty(\mathbb{R}^n)$. If $C_0(\mathbb{R}^n)$ was dense in $L^\infty(\mathbb{R}^n)$, we could find a sequence of continuous functions $f_n \in C_0(\mathbb{R}^n)$ with $\|f_n - f\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $n \to \infty$. In particular, the functions $f_n$ would converge uniformly to $f$, implying continuity of $f$, a contradiction.

**Second Step:** $C_0^\infty(\mathbb{R}^n)$ is not dense in $W^{k,\infty}(\mathbb{R}^n)$, $k \in \mathbb{N}$.

We consider the one-dimensional step function $f := \mathbb{1}_{[0,1]} : \mathbb{R} \to \mathbb{R}$ and integrate $k$ times,

$$v : \mathbb{R} \to \mathbb{R}, \quad v(\xi) := \int_0^\xi \int_0^{\xi_1} \ldots \int_0^{\xi_{k-2}} f(\xi_{k-1}) \, d\xi_{k-1} \ldots d\xi_2 d\xi_1.$$ 

Then, $v$ is $k$ times weakly differentiable\(^3\), and $v^{(k)} = f$ is not continuous.

Taking a smooth cutoff function $\psi \in C^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ in a neighborhood of $0$, we then choose

$$u(x) := v(x_1) \cdot \psi(x), \quad x \in \mathbb{R}^n.$$ 

This function, too, is an element of $C^{k-1}(\mathbb{R}^n)$, $k$ times weakly differentiable and its $k^{th}$-order weak derivative is not continuous (but bounded). Moreover, $u$ has compact support, and hence $u \in W^{k,\infty}(\mathbb{R}^n)$.

Assuming density of $C_0^\infty(\mathbb{R}^n)$ in $W^{k,\infty}(\mathbb{R}^n)$, we find (much as before) a sequence of functions $u_n \in C_0^\infty(\mathbb{R}^n)$ with $\|u_n - u\|_{W^{k,\infty}(\mathbb{R}^n)} \to 0$ as $n \to \infty$. In particular, this implies $\|\partial^\alpha u_n - \partial^\alpha u\|_{L^\infty(\mathbb{R}^n)} \to 0$ for every multiindex with $|\alpha| = k$. As above, uniform convergence and continuity of $\partial^\alpha u_n$ imply continuity of $\partial^\alpha u$, hence $u \in C^k(\mathbb{R}^n)$, a contradiction.

\[\square\]

\(^3\)More specifically, after one integration, the function $\xi \mapsto \int_0^\xi f(\xi_1) \, d\xi_1$ is continuous and piecewise linear. So, the first $k - 1$ derivatives of $v$ exist in the classical sense, whereas the $k^{th}$ derivative only exists in the weak sense.