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Bifurcation Theory

Solutions to Problem Sheet 1

Problem 1 (Bifurcation Diagrams)

Draw bifurcation diagrams for the following equations:

(a) $x^3 + 2\lambda x^2 + \lambda^3 x = 0$ $(\lambda, x \in \mathbb{R})$,

(b) $x + \sinh(\lambda x) = 0$ $(\lambda, x \in \mathbb{R})$.

Solution

Let us first note that both equations are satisfied by the trivial solution family $\{(0, \lambda) : \lambda \in \mathbb{R}\}$. We now aim to find (nontrivial) solutions $(x, \lambda) \in \mathbb{R} \times \mathbb{R}$, $x \neq 0$.

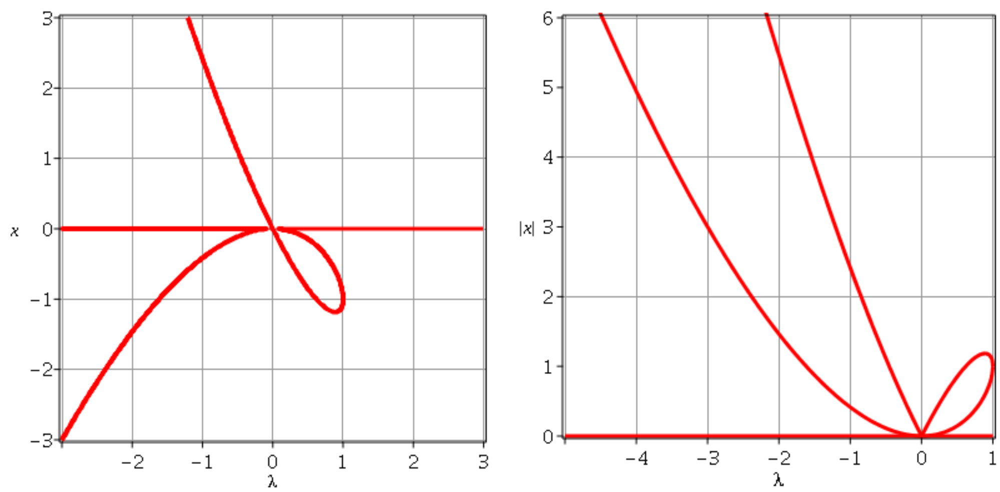
(a) We define the auxiliary function

$$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad f(x, \lambda) := x^2 + 2\lambda x + \lambda^3$$

and calculate its zeros explicitly:

$$\begin{aligned} 0 = f(x,\lambda) \quad &\Leftrightarrow \quad 0 = (x+\lambda)^2 + \lambda^3 - \lambda^2 \\ &\Leftrightarrow \quad \lambda \leq 1, \quad x \in \left\{ -\lambda \pm \sqrt{\lambda^2 - \lambda^3} \right\}. \end{aligned}$$

This yields the following bifurcation diagram which shows that $(0, 0)$ is a bifurcation point:



(b) For every $\lambda \in \mathbb{R}$, we introduce the smooth function

$$g_\lambda : \mathbb{R} \to \mathbb{R}, \quad g_\lambda(x) := x + \sinh(\lambda x).$$

We already know that $g_\lambda(0) = 0$, which corresponds to the trivial branch. Further,

$$g'_\lambda(x) = 1 + \lambda \cosh(\lambda x) \qquad (x \in \mathbb{R})$$

implies (due to $\cosh(y) > 1, y \in \mathbb{R} \setminus \{0\}$) that g_λ is strictly monotone for $\lambda \geq 0$ and $\lambda \leq -1$, so in these cases, 0 is the only zero of g_λ. In the remaining cases, we prove:

(1) *Assertion:* For $-1 < \lambda < 0$, there exists $x_\lambda > 0$ with the property that

$$g_\lambda^{-1}(\{0\}) = \{-x_\lambda, 0, x_\lambda\}.$$

Proof: Let $-1 < \lambda < 0$. As the function g_λ is odd, we only have to prove that it has exactly one positive zero x_λ.

Existence: As $\lambda < 0$, we conclude $\lim_{x\to\infty} g_\lambda(x) = -\infty$. Further, $\lambda > -1$ implies that $g'_\lambda(0) = 1 + \lambda > 0$, and hence, there exists $\delta_\lambda > 0$ with $g_\lambda(\delta_\lambda) = \max_{x>0} g_\lambda(x) > 0$. Since g_λ is continuous, Bolzano's theorem now yields the existence of a zero $x_\lambda > \delta_\lambda$.

Uniqueness: This is a consequence of the fact that g_λ is strictly concave on $(0, \infty)$ due to

$$g''_\lambda(x) = \lambda^2 \sinh(\lambda x) < 0 \qquad (x > 0).$$

(Hence, the first derivative is strictly decreasing, and g_λ has at most one critical point in $(0, \infty)$, which is $\delta_\lambda \in (0, x_\lambda)$. We infer that g_λ is strictly monotone on both $(0, \delta_\lambda)$ and (δ_λ, ∞), which proves the assertion.) ■

(2) *Assertion:* The mapping $\lambda \mapsto x_\lambda$ $(-1 < \lambda < 0)$ is continuous.

Proof: This is a consequence of the Implicit Function Theorem, which can be applied since

$$g_\lambda(x_\lambda) = 0, \qquad g'_\lambda(x_\lambda) < 0$$

(where the latter inequality results from strict concavity, see above). ■

(3) *Assertion:* We have the following one-sided limits:

$$\lim_{\lambda \to -1_+} x_\lambda = 0, \qquad \lim_{\lambda \to 0_-} x_\lambda = \infty.$$

Proof: For $\lambda \in (-1, 0)$, we estimate by power series expansion:

$$x_\lambda = -\sinh(\lambda x_\lambda) = \sinh(|\lambda| x_\lambda) \geq |\lambda| x_\lambda + \frac{1}{6}|\lambda|^3 x_\lambda^3,$$

which yields

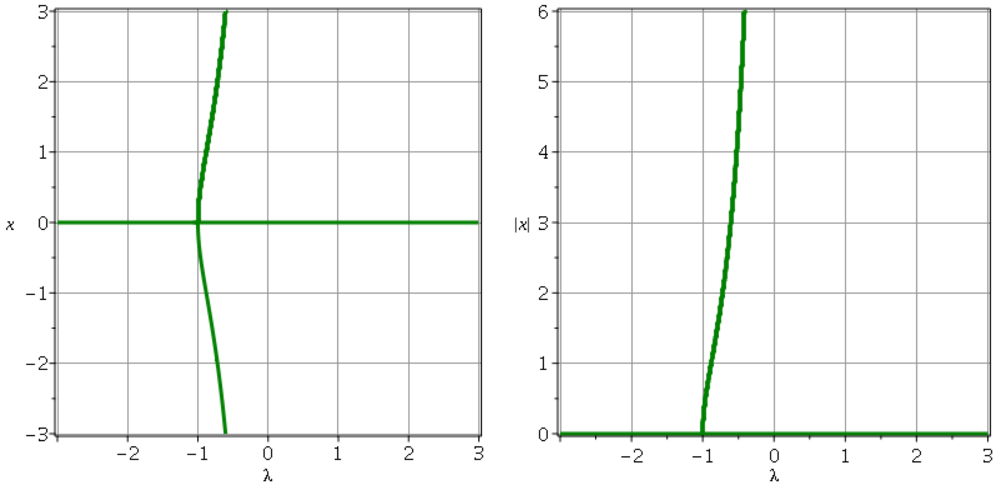
$$x_\lambda^2 \le \frac{6}{|\lambda|^3}(1-|\lambda|) \to 0 \qquad (\lambda \to -1,\ \ \lambda > -1).$$

On the other hand, we recall that $g_\lambda'(x_\lambda) < 0$ and obtain by inserting $g_\lambda(x_\lambda) = 0$

$$0 > 1 - |\lambda|\cosh(\lambda x_\lambda) = 1 - |\lambda|\sqrt{1+\sinh^2(\lambda x_\lambda)} = 1 - |\lambda|\sqrt{1+x_\lambda^2}.$$

This can only hold in the limit $\lambda \to 0$, $\lambda < 0$ if, at the same time, $x_\lambda \to \infty$. ■

This yields the following bifurcation diagram:



Problem 2 (Bifurcation with respect to different topologies)

Consider the function $u_1 : \mathbb{R} \to \mathbb{R},\ x \mapsto \frac{\sqrt{2}}{\cosh(x)}$.

(a) Prove that $u_1 \in W^{2,q}(\mathbb{R})$ for all $q \in [1, \infty]$.

(b) Prove that u_1 solves the ODE $-u'' + u - u^3 = 0$ on $\mathbb{R}$.

(c) Find a nontrivial family of solutions $\mathcal{T} := \{(u_\lambda, \lambda) :\ \lambda > 0\}$ of

$$\begin{cases} -u'' + \lambda u - u^3 = 0, \\ u \in W^{2,q}(\mathbb{R}) \end{cases} \tag{1}$$

and, for each $q \in [1, \infty]$, decide whether it bifurcates from the trivial branch at $(0, 0)$ with respect to $\|\cdot\|_{W^{2,q}(\mathbb{R})}$.

Solution

(a) Clearly, u_1 is a smooth function. For $x \in \mathbb{R}$, we have (with $\sinh^2(x) + 1 = \cosh^2(x)$)

$$\begin{aligned} u_1'(x) &= -\frac{\sqrt{2}}{\cosh^2(x)} \cdot \sinh(x), \\ u_1''(x) &= \frac{2\sqrt{2}}{\cosh^3(x)} \cdot \sinh^2(x) - \frac{\sqrt{2}}{\cosh^2(x)} \cdot \cosh(x) = \frac{\sqrt{2}}{\cosh^3(x)} \cdot \left[2\sinh^2(x) - \cosh^2(x)\right] \\ &= \frac{\sqrt{2}}{\cosh^3(x)} \cdot \left[\cosh^2(x) - 2\right] = u_1(x) - u_1(x)^3. \qquad (\diamondsuit) \end{aligned}$$

As $u_1 \in C^\infty(\mathbb{R})$, weak derivatives of second (in fact, of every) order exist and agree with the classical ones computed above.

Let $q \in [1, \infty]$. It remains to check integrability, i.e. whether $u_1, u_1', u_1'' \in L^q(\mathbb{R})$. We exploit that, for $x \in \mathbb{R}$,

$$\cosh(x) = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} \geq \frac{1}{2}\mathrm{e}^{|x|} \quad \text{and} \quad |\sinh(x)| = \frac{|\mathrm{e}^x - \mathrm{e}^{-x}|}{2} \leq \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2} = \cosh(x),$$

and estimate as follows:

$$\begin{aligned} |u_1(x)| &= \frac{\sqrt{2}}{\cosh(x)} \leq 2\sqrt{2}\,\mathrm{e}^{-|x|}, \\ |u_1'(x)| &= \frac{\sqrt{2}}{\cosh^2(x)} \cdot \sinh(|x|) \leq \frac{\sqrt{2}}{\cosh(x)} \leq 2\sqrt{2}\,\mathrm{e}^{-|x|}, \\ |u_1''(x)| &= \frac{\sqrt{2}}{\cosh^3(x)} \cdot |\cosh^2(x) - 2| \leq \frac{\sqrt{2}}{\cosh(x)} + \left(\frac{\sqrt{2}}{\cosh(x)}\right)^3 \leq 2\sqrt{2}\,\mathrm{e}^{-|x|} + 16\sqrt{2}\,\mathrm{e}^{-3|x|} \\ &\leq 18\sqrt{2}\,\mathrm{e}^{-|x|}. \end{aligned}$$

Since $\mathrm{e}^{-|\cdot|} \in L^q(\mathbb{R})$, we conclude that $u_1 \in W^{2,q}(\mathbb{R})$ as claimed.

(b) This has been shown in the course of (a), equation ($\diamondsuit$).

(c) <u>First Step</u>: *We construct a branch of nontrivial solutions* (u_λ, λ) *(*$\lambda > 0$*) by scaling.*

For $\lambda > 0$ and $x \in \mathbb{R}$, we define

$$u_\lambda(x) := \sqrt{\lambda} \cdot u_1(\sqrt{\lambda}\, x) = \frac{\sqrt{2\lambda}}{\cosh(\sqrt{\lambda}\, x)} \qquad (x \in \mathbb{R}, \lambda > 0).$$

As in the first step, we see that $u_\lambda \in W^{2,q}(\mathbb{R})$ for every $q \in [1, \infty]$ and note that $u_\lambda \neq 0$. Moreover, u_λ is a smooth function and we have for $x \in \mathbb{R}$

$$-u_\lambda''(x) + \lambda u_\lambda(x) - u_\lambda^3(x) = \lambda^{\frac{3}{2}} \cdot \Big(- u_1''(\sqrt{\lambda}\, x) + u_1(\sqrt{\lambda}\, x) - u_1(\sqrt{\lambda}\, x)^3 \Big) = 0.$$

Hence, for all $\lambda > 0$, $u_\lambda \in W^{2,q}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ is a classical solution to

$$-u'' + \lambda u = u^3 \qquad \text{in } \mathbb{R}.$$

So we have found a family $\mathcal{T} := \{(u_\lambda, \lambda) : \ \lambda > 0\}$ of nontrivial solutions to (1).

<u>Second Step</u>: *We discuss whether* $\mathcal{T}$ *bifurcates from* $(0,0)$.

To this end, we fix $q \in [1, \infty]$, calculate the norms $\left\| u_\lambda^{(j)} \right\|_{L^q(\mathbb{R})}$, $j = 0, 1, 2$, and discuss the limit $\lambda \to 0$. We have for $\lambda > 0$ and $x \in \mathbb{R}$

$$u_\lambda(x) = \sqrt{\lambda} \cdot u_1(\sqrt{\lambda}\, x), \quad u_\lambda'(x) = \lambda \cdot u_1'(\sqrt{\lambda}\, x), \quad u_\lambda''(x) = \lambda\sqrt{\lambda} \cdot u_1''(\sqrt{\lambda}\, x).$$

Hence, for $j = 0, 1, 2$,

$$\left\| u_\lambda^{(j)} \right\|_{L^\infty(\mathbb{R})} = \lambda^{\frac{j+1}{2}} \left\| u_1^{(j)} \right\|_{L^\infty(\mathbb{R})} \overset{\lambda \to 0}{\longrightarrow} 0,$$

and for $1 \leq q < \infty$,

$$\begin{aligned} \left\| u_\lambda^{(j)} \right\|_{L^q(\mathbb{R})}^q &= \int_{\mathbb{R}} |u_\lambda^{(j)}(x)|^q \,\mathrm{d}x = \int_{\mathbb{R}} |\lambda^{\frac{j+1}{2}} \cdot u_1^{(j)}(\sqrt{\lambda}\, x)|^q \,\mathrm{d}x \overset{y = \sqrt{\lambda}\, x}{=} \lambda^{\frac{qj+q-1}{2}} \int_{\mathbb{R}} |u_1^{(j)}(y)|^q \,\mathrm{d}y \\ &= \lambda^{\frac{qj+q-1}{2}} \left\| u_1^{(j)} \right\|_{L^q(\mathbb{R})}^q \overset{\lambda \to 0}{\longrightarrow} \begin{cases} 0, & 1 < q < \infty \text{ or } j = 1, 2, \\ \|u_1\|_{L^q(\mathbb{R})}^q > 0, & q = 1 \text{ and } j = 0. \end{cases} \end{aligned}$$

We conclude that

$$\lim_{\lambda \searrow 0} \|u_\lambda\|_{W^{2,q}(\mathbb{R})} = 0 \quad \Leftrightarrow \quad 1 < q \leq \infty,$$

and bifurcation from the trivial branch with respect to $\| \cdot \|_{W^{2,q}(\mathbb{R})}$ occurs if and only if $q \in (1, \infty]$.

□

Problem 3 (Density of $C_0^\infty(\mathbb{R}^n)$)

For $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n), g \in L^1(\mathbb{R}^n)$, we define the *convolution* $f * g \in L^p(\mathbb{R}^n)$

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y.$$

Then, *Young's inequality* states that $\|f * g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}$.

(a) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$ and for $y \in \mathbb{R}^n$, define the translation $(\tau_y f)(x) := f(x-y)$, $x \in \mathbb{R}^n$. Argue that $\tau_y f \in L^p(\mathbb{R}^n)$ and prove

$$\|f - \tau_y f\|_{L^p(\mathbb{R}^n)} \to 0 \qquad \text{as } y \to 0.$$

(b) Let $1 \leq p < \infty$, $f \in L^p(\mathbb{R}^n)$ and $\psi_\varepsilon(x) := \varepsilon^{-n}\psi\left(\frac{1}{\varepsilon}x\right)$ for $\varepsilon > 0$ and some $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\,\mathrm{d}x = 1$ and $\psi \geq 0$. Prove that

$$\|\psi_\varepsilon * f - f\|_{L^p(\mathbb{R}^n)} \to 0 \quad (\varepsilon \searrow 0).$$

(c) Conclude that $C_0^\infty(\mathbb{R}^n)$ is dense both in $L^p(\mathbb{R}^n)$ and in $W^{k,p}(\mathbb{R}^n)$ for $1 \leq p < \infty$, $k \in \mathbb{N}$.

(d) Argue that $C_0^\infty(\mathbb{R}^n)$ is dense neither in $L^\infty(\mathbb{R}^n)$ nor in $W^{k,\infty}(\mathbb{R}^n)$ for any $k \in \mathbb{N}$.

Some Hints:
For parts (a) and (b), recall that $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. (Proof in the Problem Class.) In part (c), you can use without proof that $f \in L^p(\mathbb{R}^n), g \in C_0^\infty(\mathbb{R}^n)$ implies that $f * g \in L^p(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ and that, if additionally f has compact support, $f * g \in C_0^\infty(\mathbb{R}^n)$.

Solution

(a) Let $y \in \mathbb{R}^n$. As a direct consequence of the invariance of the Lebesgue measure under translations, $\tau_y f$ is measurable and the coordinate transformation $\mathbb{R}^n \to \mathbb{R}^n$, $x - y \mapsto x$ yields $\|\tau_y f\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}$.

First Step: We prove the convergence property for a compactly supported continuous function $g \in C_0(\mathbb{R}^n) \subseteq L^p(\mathbb{R}^n)$.

As g is continuous with compact support, it is in particular bounded, and we choose $C_g > 0$ with $|g(x)| \leq C_g$ for all $x \in \mathbb{R}^n$. Moreover, we denote the (compact) support of g by $K_g := \operatorname{spt} g$ and introduce the Minkowski sum $K_g(\varrho) := K_g + \overline{B}_\varrho(0)$ for $\varrho > 0$. We aim to prove

$$\left(\int_{\mathbb{R}^n} |g(x-y) - g(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \to 0 \qquad \text{as } y \to 0$$

by dominated convergence. By continuity of g, we have pointwise convergence

$$\big|g(x-y)-g(x)\big|^p \to 0$$

as $y \to 0$ for every fixed $x \in \mathbb{R}^n$. Moreover, considering $y \in \mathbb{R}^n$ with $|y| \leq 1$ only, we have the estimate

$$|g(x-y)-g(x)|^p \leq (2C_g)^p \cdot \mathbb{1}_{K_g(1)}(x)$$

where the right-hand side is integrable due to compactness of $K_g(1)$. Thus, an application of dominated convergence proves the assertion.

<u>Second Step</u>: *We now consider $f \in L^p(\mathbb{R}^n)$.*

Let $\eta > 0$. By the density of compactly supported continuous functions in $L^p(\mathbb{R}^n)$, we find $g \in C_0(\mathbb{R}^n)$ with the property that $\|f-g\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{3}$. Moreover, the first step yields $\delta > 0$ with the property that $\|\tau_y g - g\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{3}$ for every $y \in \mathbb{R}^n$, $|y| < \delta$. The assertion now follows from the triangle inequality; for $y \in \mathbb{R}^n$, $|y| < \delta$ we have

$$\begin{aligned}\|\tau_y f - f\|_{L^p(\mathbb{R}^n)} &\leq \|\tau_y(f-g)\|_{L^p(\mathbb{R}^n)} + \|\tau_y g - g\|_{L^p(\mathbb{R}^n)} + \|g-f\|_{L^p(\mathbb{R}^n)} \\ &= \|\tau_y g - g\|_{L^p(\mathbb{R}^n)} + 2\,\|g-f\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{3} + 2\cdot\frac{\eta}{3} = \eta.\end{aligned}$$

□

(b) For $\varepsilon > 0$, we have

$$\int_{\mathbb{R}^n} \psi_\varepsilon(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \varepsilon^{-n}\psi\left(\frac{1}{\varepsilon}x\right)\,\mathrm{d}x \overset{x=\varepsilon y}{=} \int_{\mathbb{R}^n} \psi(y)\,\mathrm{d}y = 1.$$

We first use commutativity of the convolution ($\psi_\varepsilon * f = f * \psi_\varepsilon$), then this result, afterwards Hölder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$, finally Tonelli's theorem and obtain

$$\begin{aligned}\|\psi_\varepsilon * f - f\|^p_{L^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left|\left(\int_{\mathbb{R}^n} \psi_\varepsilon(y) f(x-y)\,\mathrm{d}y\right) - f(x)\right|^p \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \psi_\varepsilon(y)\big(f(x-y)-f(x)\big)\,\mathrm{d}y\right|^p \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \psi_\varepsilon(y)^{\frac{1}{p'}} \cdot \psi_\varepsilon(y)^{\frac{1}{p}}\big(f(x-y)-f(x)\big)\,\mathrm{d}y\right|^p \mathrm{d}x \\ &\leq \left(\int_{\mathbb{R}^n} |\psi_\varepsilon(y)^{\frac{1}{p'}}|^{p'}\,\mathrm{d}y\right)^{\frac{p}{p'}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\psi_\varepsilon(y)^{\frac{1}{p}}\big(f(x-y)-f(x)\big)\right|^p \mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \psi_\varepsilon(y)\big|f(x-y)-f(x)\big|^p\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \psi_\varepsilon(y)\left(\int_{\mathbb{R}^n} \big|f(x-y)-f(x)\big|^p\,\mathrm{d}x\right)\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \psi_\varepsilon(y)\,\|\tau_y f - f\|^p_{L^p(\mathbb{R}^n)}\,\mathrm{d}y \qquad (2)\end{aligned}$$

Now, we let $\eta > 0$. By virtue of (a), we can choose $\delta > 0$ with the property that, for $y \in \mathbb{R}^n$ with $|y| \leq \delta$,

$$\|\tau_y f - f\|^p_{L^p(\mathbb{R}^n)} < \eta. \tag{3}$$

Moreover, by definition of ψ_ε, we have that $\psi(x) = 0 \Leftrightarrow \psi_\varepsilon(\varepsilon x) = 0$ and thus

$$\operatorname{spt} \psi_\varepsilon = \{\varepsilon x : \ x \in \operatorname{spt} \psi\} = \varepsilon \cdot \operatorname{spt} \psi.$$

We therefore find $\varepsilon_0 > 0$ with the property that, for every $\varepsilon \in (0, \varepsilon_0)$,

$$\operatorname{spt} \psi_\varepsilon \subseteq B_\delta(0). \tag{4}$$

With that, we continue the estimate (2) and conclude for $\varepsilon \in (0, \varepsilon_0)$

$$\begin{aligned} \|\psi_\varepsilon * f - f\|^p_{L^p(\mathbb{R}^n)} &\leq \int_{\mathbb{R}^n} \psi_\varepsilon(y) \, \|\tau_y f - f\|^p_{L^p(\mathbb{R}^n)} \, \mathrm{d}y \\ &\overset{(4)}{=} \int_{B_\delta(0)} \psi_\varepsilon(y) \, \|\tau_y f - f\|^p_{L^p(\mathbb{R}^n)} \, \mathrm{d}y \overset{(3)}{<} \eta \int_{B_\delta(0)} \psi_\varepsilon(y) \, \mathrm{d}y \leq \eta. \end{aligned}$$

This closes the proof of part (b). □

(c) We let $\psi, \psi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$, $\varepsilon > 0$, as in part (b). (For the existence of such a function, we refer to the lecture *Analysis 2*.)

<u>*First Step*</u>*: $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.*

We let $f \in L^p(\mathbb{R}^n)$ and $\eta > 0$.

As in the proof of part (a), we can choose $g \in C_0(\mathbb{R}^n)$ with $\|f - g\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{2}$, and it suffices to find $h \in C_0^\infty(\mathbb{R}^n)$ with $\|g - h\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{2}$. The hint implies that the convolutions satisfy $\psi_\varepsilon * g \in C_0^\infty(\mathbb{R}^n)$, and part (b) gives

$$\|\psi_\varepsilon * g - g\|_{L^p(\mathbb{R}^n)} \to 0 \qquad (\varepsilon \searrow 0).$$

Hence, choosing $h := \psi_\varepsilon * g$ with $\varepsilon > 0$ so small that $\|\psi_\varepsilon * g - g\|_{L^p(\mathbb{R}^n)} < \frac{\eta}{2}$, we have found a suitable function $h \in C_0^\infty(\mathbb{R}^n)$.

<u>*Second Step*</u>*: $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$, $1 \leq p < \infty$ and $k \in \mathbb{N}$.*

We introduce the space $W^{k,p}_{\mathrm{cpt}}(\mathbb{R}^n) := \{u \in W^{k,p}(\mathbb{R}^n) : \ \operatorname{spt}(u) \ \text{is compact}\}$[1].

Second step, first part: $W^{k,p}_{\mathrm{cpt}}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$.

We consider $u \in W^{k,p}(\mathbb{R}^n)$ and introduce a smooth cutoff function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\mathbb{1}_{B_1(0)} \leq \chi \leq \mathbb{1}_{B_2(0)}$. We define $C := \max_{|\alpha| \leq k} \|\partial^\alpha \chi\|_\infty$ and, for $R > 0$ and $x \in \mathbb{R}^n$, $\chi_R(x) := \chi\left(\frac{1}{R} x\right)$.

[1]This does not take into account that $W^{k,p}(\mathbb{R}^n)$ actually contains equivalence classes of functions. Strictly speaking, we demand that u has a representative whose support is compact, and henceforth consider that representative.

Let $\eta > 0$. By monotone convergence, we find $R > 0$ with the property that, for every multiindex with $|\alpha| \le k$,

$$\|\partial^\alpha u\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \le \frac{\eta}{2 \cdot \max\{C, 1\}}$$

and, additionally, $R \ge 1$. We note that $u \cdot \chi_R \in W^{k,p}_{\mathrm{cpt}}(\mathbb{R}^n)$ and estimate for an arbitrary multiindex with $|\alpha| \le k$

$$\begin{aligned}
\|\partial^\alpha(u \cdot \chi_R) - \partial^\alpha u\|_{L^p(\mathbb{R}^n)} &\le \|(\chi_R - 1) \cdot \partial^\alpha u\|_{L^p(\mathbb{R}^n)} + \|u \cdot \partial^\alpha \chi_R\|_{L^p(\mathbb{R}^n)} \\
&= \|(\chi_R - 1) \cdot \partial^\alpha u\|_{L^p(\mathbb{R}^n \setminus B_R(0))} + \|u \cdot \partial^\alpha \chi_R\|_{L^p(B_{2R}(0) \setminus B_R(0))} \\
&\le \|\partial^\alpha u\|_{L^p(\mathbb{R}^n \setminus B_R(0))} + \frac{C}{R^{|\alpha|}} \cdot \|u\|_{L^p(\mathbb{R}^n \setminus B_R(0))} \\
&\le \frac{\eta}{2} + C \cdot \frac{\eta}{2C} = \eta
\end{aligned}$$

where we have used that $\mathbb{1}_{B_R(0)} \le \chi_R \le \mathbb{1}_{B_{2R}(0)}$ and $\partial^\alpha \chi_R(x) = R^{-|\alpha|}(\partial^\alpha \chi)\left(\frac{1}{R}x\right)$. This proves the first part.

Second step, second part: $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{k,p}_{\mathrm{cpt}}(\mathbb{R}^n)$ (in the topology of $W^{k,p}(\mathbb{R}^n)$).

We let $u \in W^{k,p}_{\mathrm{cpt}}(\mathbb{R}^n)$ and $\eta > 0$. Taking $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ as in part (a), the hint now gives[2] that $\psi_\varepsilon * u \in C_0^\infty(\mathbb{R}^n)$. For every multiindex with $|\alpha| \le k$, part (b) implies

$$\|\partial^\alpha(\psi_\varepsilon * u) - \partial^\alpha u\|_{L^p(\mathbb{R}^n)} = \|\psi_\varepsilon * \partial^\alpha u - \partial^\alpha u\|_{L^p(\mathbb{R}^n)} \to 0 \qquad (\varepsilon \searrow 0).$$

Thus, taking $\varepsilon > 0$ sufficiently small, we find that

$$\|\psi_\varepsilon * u - u\|_{W^{k,p}(\mathbb{R}^n)} < \eta,$$

which proves the assertion. Combining the first and the second part of step 2, the proof is complete. □

[2]Note that, if only $u \in W^{k,p}(\mathbb{R}^n)$, we would find $\psi_\varepsilon * u \in C^\infty(\mathbb{R}^n)$ - compact support is lost in the process of smoothing via convolution.

(d) *First Step: $C_0^\infty(\mathbb{R}^n)$ is not dense in $L^\infty(\mathbb{R}^n)$.*

We prove more generally that $C_0(\mathbb{R}^n)$ is not dense in $L^\infty(\mathbb{R}^n)$.

We consider the step function $f := \mathbb{1}_{B_1(0)}$, which is an element of $L^\infty(\mathbb{R}^n)$. If $C_0(\mathbb{R}^n)$ was dense in $L^\infty(\mathbb{R}^n)$, we could find a sequence of continuous functions $f_n \in C_0(\mathbb{R}^n)$ with $\|f_n - f\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $n \to \infty$. In particular, the functions f_n would converge uniformly to f, implying continuity of f, a contradiction.

Second Step: $C_0^\infty(\mathbb{R}^n)$ is not dense in $W^{k,\infty}(\mathbb{R}^n)$, $k \in \mathbb{N}$.

We consider the one-dimensional step function $f := \mathbb{1}_{[0,1]} : \ \mathbb{R} \to \mathbb{R}$ and integrate k times,

$$v : \ \mathbb{R} \to \mathbb{R}, \quad v(\xi) := \int_0^\xi \int_0^{\xi_1} \dots \int_0^{\xi_{k-2}} f(\xi_{k-1}) \, \mathrm{d}\xi_{k-1} \dots \mathrm{d}\xi_2 \mathrm{d}\xi_1.$$

Then, v is k times weakly differentiable[3], and $v^{(k)} = f$ is not continuous.

Taking a smooth cutoff function $\psi \in C^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ in a neighborhood of 0, we then choose

$$u(x) := v(x_1) \cdot \psi(x), \qquad x \in \mathbb{R}^n.$$

This function, too, is an element of $C^{k-1}(\mathbb{R}^n)$, k times weakly differentiable and its k^{th}-order weak derivative is not continuous (but bounded). Moreover, u has compact support, and hence $u \in W^{k,\infty}(\mathbb{R}^n)$.

Assuming density of $C_0^\infty(\mathbb{R}^n)$ in $W^{k,\infty}(\mathbb{R}^n)$, we find (much as before) a sequence of functions $u_n \in C_0^\infty(\mathbb{R}^n)$ with $\|u_n - u\|_{W^{k,\infty}(\mathbb{R}^n)} \to 0$ as $n \to \infty$. In particular, this implies $\|\partial^\alpha u_n - \partial^\alpha u\|_{L^\infty(\mathbb{R}^n)} \to 0$ for every multiindex with $|\alpha| = k$. As above, uniform convergence and continuity of $\partial^\alpha u_n$ imply continuity of $\partial^\alpha u$, hence $u \in C^k(\mathbb{R}^n)$, a contradiction.

$\square$

[3] More specifically, after one integration, the function $\xi \mapsto \int_0^\xi f(\xi_1) \, \mathrm{d}\xi_1$ is continuous and piecewise linear. So, the first $k-1$ derivatives of v exist in the classical sense, whereas the k^{th} derivative only exists in the weak sense.