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# Bifurcation Theory

## Solutions to Problem Sheet 2

### Problem 4 (On the Energy Method)

Find such continuous functions $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $g(0, \mu) = 0$ for all $\mu \in \mathbb{R}$ that

$$\begin{cases} -u'' = g(u, \mu) & \text{in } (0, 1), \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

has the asserted properties.

**(a)** There is such $\mu_0 \in \mathbb{R}$ that (1) does not admit a nontrivial solution for $\mu \geq \mu_0$.

**(b)** There exist $\mu_0 \in \mathbb{R}$ and sequences $(u_n, \mu_n)_{n\in\mathbb{N}}, (\tilde{u}_n, \tilde{\mu}_n)_{n\in\mathbb{N}} \subseteq C^2([0, 1]) \times \mathbb{R}$ of solutions of (1) which bifurcate from the trivial family at $(0, \mu_0)$ and satisfy

$$u_n, \tilde{u}_n \neq 0, \quad \mu_n < \mu_0 < \tilde{\mu}_n \quad \text{for all } n \in \mathbb{N}.$$

**(c)** We have $g \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist $\alpha_0 > 0$, $j \in \mathbb{N}_0$, $\lambda_0 \in \mathbb{R}$ with

$$g(-z, \mu_0) = -g(z, \mu_0), \;\; g(z, \mu_0)z > 0 \quad \text{for } 0 < |z| < \alpha_0;$$
$$\int_0^\alpha (G(\alpha, \mu_0) - G(z, \mu_0))^{-\frac{1}{2}} \, \mathrm{d}z < \infty \quad \text{for } 0 < \alpha < \alpha_0;$$
$$g_z(0, \mu_0) = \pi^2 (j + 1)^2.$$

According to Theorem II.3 from the lecture, the *transversality condition* $g_{z\mu}(0, \mu_0) \neq 0$ implies that $j$-nodal solutions of (1) bifurcate from $(0, \mu_0)$ in $C^2([0, 1])$. Demonstrate that bifurcation need not occur if the transversality condition is violated.

### Solution

**(a)** For arbitrary $\mu_0 \in \mathbb{R}$, consider $g(z, \mu) := (\mu_0 - \mu)z$.

Then a direct calculation shows that, for $\mu \geq \mu_0$, the differential equation in (1) has a solution of the form $u(t) = \alpha \cosh(\sqrt{\mu - \mu_0}t) + \beta \sinh(\sqrt{\mu - \mu_0}t)$ with coefficients $\alpha, \beta \in \mathbb{R}$. The homogeneous boundary conditions now give

$$0 = u(0) = \alpha, \qquad 0 = u(1) = \alpha \cosh(\sqrt{\mu - \mu_0}) + \beta \sinh(\sqrt{\mu - \mu_0}) = \beta \sinh(\sqrt{\mu - \mu_0}),$$

hence $\alpha = \beta = 0$ and $u \equiv 0$.

**(b)** Consider, as a slight modification of the pendulum equation discussed in the lecture, $g(z,\mu) = (\lambda_j + (\mu - \lambda_j)^2)\sin(z)$ where $\lambda_j = ((j+1)\pi)^2$ for some $j \in \mathbb{N}_0$, and consider $\mu_0 := \lambda_j$.

We know from the lecture that, for $-u'' = \lambda \sin(u), u(0) = u(1) = 0$, $j$-nodal solutions bifurcate at the point $(0, \lambda_j)$, parametrized as $(\alpha, \lambda_j(\alpha))_{0<\alpha<\alpha_0}$ with $\lambda_j(\alpha) \searrow \lambda_j$ as $\alpha \searrow 0$. With $g$ chosen as above, we find two parameters $\lambda$ corresponding to each value of $\alpha$ via

$$\lambda_j(\alpha) = \lambda_j + (\mu - \lambda_j)^2 \quad \Leftrightarrow \quad \mu = \lambda_j \pm \sqrt{\lambda_j(\alpha) - \lambda_j},$$

which yields to distinct families of bifurcating $j$-nodal solutions of (1) parametrized as

$$\left(\alpha, \lambda_j + \sqrt{\lambda_j(\alpha) - \lambda_j}\right)_{0<\alpha<\alpha_0}, \quad \left(\alpha, \lambda_j - \sqrt{\lambda_j(\alpha) - \lambda_j}\right)_{0<\alpha<\alpha_0}.$$

In the context of this problem, we choose $\alpha_n \searrow 0$, $0 < \alpha_n < \alpha_0$ and let

$$\mu_n := \lambda_j - \sqrt{\lambda_j(\alpha_n) - \lambda_j}, \quad \tilde{\mu}_n := \lambda_j + \sqrt{\lambda_j(\alpha_n) - \lambda_j}, \quad u_n = \tilde{u}_n$$

with the suitable $j$-nodal solution of $-u'' = \lambda_j(\alpha_n)\sin(u), u(0) = u(1) = 0$ with $\|u_n\| = \alpha_n$.

**(c)** Again, we modify the pendulum equation. We introduce $g(z,\mu) = (\lambda_j - (\mu - \lambda_j)^2)\sin(z)$ where $\lambda_j = ((j+1)\pi)^2$ for some $j \in \mathbb{N}_0$, and consider $\mu_0 := \lambda_j$.

However, using the notation from part (b), the equation

$$\lambda_j(\alpha) = \lambda_j - (\mu - \lambda_j)^2$$

does not have solutions, and the bifurcation diagram for $-u'' = \lambda \sin(u), u(0) = u(1) = 0$ from the lecture reveals that there is no bifurcation of (1) at $(0, \lambda_j)$.

As in the lecture, we see that all conditions of Theorem II.3 are satisfied, in particular

$$g_z(0, \lambda_j) = \lambda_j \cdot \cos(0) = \pi^2 (j+1)^2,$$

but for the transversality condition

$$g_{z\mu}(0, \lambda_j) = -2(\lambda_j - \lambda_j) \cdot \cos(1) = 0.$$

**Problem 5 (An application of the Energy Method)**

Consider the boundary value problem

$$\begin{cases} u'' + \lambda(u - u^3) = 0 & \text{in } (0,\pi), \\ u(0) = u(\pi) = 0. \end{cases} \tag{2}$$

Prove that, for $k \in \mathbb{N}$, nontrivial $(k-1)$-nodal solutions of (2) bifurcate from the trivial branch at the point $\lambda_k = k^2$.

**Solution**

We intend to apply Theorem II.3 and define

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad g(z,\lambda) := \lambda(z - z^3).$$

Then $g \in C^\infty(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g(0,\lambda) = 0$ for every $\lambda \in \mathbb{R}$. Fixing $\lambda > 0$, we check the assumptions of Theorem II.3:

(i) For all $z \in \mathbb{R}$, we have $g(-z,\lambda) = -g(z,\lambda)$.

(ii) $g(z,\lambda)\, z = \lambda(z^2 - z^4) = \lambda z^2(1 - z^2) > 0$ holds for $z \in \mathbb{R}$ with $0 < |z| < 1$.

(iii) We let $G(z,\lambda) := \int_0^z g(s,\lambda)\,\mathrm{d}s = \frac{1}{4}\lambda(2z^2 - z^4)$ be the primitive of $g(\cdot,\lambda)$. For $\alpha \in (0,1)$, we calculate

$$\begin{aligned} \int_0^\alpha (G(\alpha,\lambda) - G(z,\lambda))^{-\frac{1}{2}}\,\mathrm{d}z &= \frac{2}{\sqrt{\lambda}} \int_0^\alpha (2\alpha^2 - \alpha^4 - 2z^2 + z^4)^{-\frac{1}{2}}\,\mathrm{d}z \\ &= \frac{2}{\sqrt{\lambda}} \int_0^\alpha ((1-z^2)^2 - (1-\alpha^2)^2)^{-\frac{1}{2}}\,\mathrm{d}z. \end{aligned}$$

In order to prove finiteness of this integral, we use the mean value theorem to find $\xi_z \in (z,\alpha)$ with

$$\begin{aligned} (1-z^2)^2 - (1-\alpha^2)^2 = (z-\alpha)\cdot \frac{\mathrm{d}}{\mathrm{d}\xi}\Big|_{\xi=\xi_z} (1-\xi^2)^2 &= (\alpha - z)\cdot 4\xi_z(1-\xi_z^2) \\ &\geq (\alpha - z)\cdot 4z(1-\alpha^2), \end{aligned}$$

and then estimate as follows:

$$\begin{aligned} \int_0^\alpha ((1-z^2)^2 - (1-\alpha^2)^2)^{-\frac{1}{2}}\,\mathrm{d}z &\leq \int_0^\alpha ((\alpha - z)\cdot 4z(1-\alpha^2))^{-\frac{1}{2}}\,\mathrm{d}z \\ &= \int_0^{\frac{\alpha}{2}} \frac{\mathrm{d}z}{\sqrt{2(\alpha - z)}\sqrt{z}\sqrt{1-\alpha^2}} + \int_{\frac{\alpha}{2}}^\alpha \frac{\mathrm{d}z}{\sqrt{\alpha - z}\sqrt{2z}\sqrt{1-\alpha^2}} \\ &\leq \int_0^{\frac{\alpha}{2}} \frac{\mathrm{d}z}{\sqrt{2\alpha}\sqrt{z}\sqrt{1-\alpha^2}} + \int_{\frac{\alpha}{2}}^\alpha \frac{\mathrm{d}z}{\sqrt{\alpha - z}\sqrt{2\alpha}\sqrt{1-\alpha^2}} \\ &= \frac{2}{\sqrt{2\alpha}\sqrt{1-\alpha^2}} \int_0^{\frac{\alpha}{2}} \frac{\mathrm{d}z}{\sqrt{z}} = \frac{2}{\sqrt{2\alpha}\sqrt{1-\alpha^2}} \cdot \sqrt{2\alpha} < \infty. \end{aligned}$$

As $g \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, Theorem II.3 (with $T = \pi$ and $\alpha_0 = 1$) gives the following necessary condition for bifurcation of $(k-1)$-nodal solutions from the trivial family at a point $(0, \lambda_k)$:

$$\partial_z g(0, \lambda_k) = \left(\frac{\pi k}{T}\right)^2, \qquad (\clubsuit)$$

and equivalently $\lambda_k = k^2$. Further, since the transversality condition (ii) of Theorem II.3,

$$\partial_\lambda \partial_z g(0, \lambda_k) = 1 \neq 0$$

holds, the condition $(\clubsuit)$ is sufficient for bifurcation, and hence the proof is complete.

$\square$

## Problem 6 (A problem without periodic Solutions)

Let $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $g(0, \lambda) = 0$ ($\lambda \in \mathbb{R}$). Consider $b \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ with $b(x, \lambda) \neq 0$ for all $x, \lambda \in \mathbb{R}$.

Prove that the differential equation

$$-u'' + b(x, \lambda)\, u' = g(u, \lambda) \qquad \text{in } \mathbb{R}, \tag{3}$$

does not admit a periodic solution in $C^2(\mathbb{R})$ unless it is constant.

## Solution

First, we note that by assumption, $b$ is continuous and does not have any zero, so $b$ is either negative or positive on all of $\mathbb{R} \times \mathbb{R}$.

We assume that $u \in C^2(\mathbb{R})$ is a periodic solution of (3). We introduce

$$E : \mathbb{R} \to \mathbb{R}, \quad E(t) := \frac{1}{2}\left[u'(t)\right]^2 + G(u(t), \lambda)$$

where $G(z, \lambda) := \int_0^z g(s, \lambda)\, \mathrm{d}s$ for $\lambda, z \in \mathbb{R}$. Then, $E \in C^1(\mathbb{R})$, and for $t \in \mathbb{R}$

$$E'(t) = u'(t) \cdot (u''(t) + g(u(t), \lambda)) = b(t, \lambda)\left[u'(t)\right]^2 \qquad (*)$$

where we have inserted the differential equation in the last step. Since $b$ does not change sign, this implies that $E$ is a monotone function. As $u$ is periodic, so is $E$, and we conclude that $E$ is constant.

Hence, $E' \equiv 0$, and by $(*)$, $u' \equiv 0$. So $u$ is constant, and the assertion is proved.

□

## Problem 7 (Variational Methods)

Let $n \in \mathbb{N}$. We consider the $n$-dimensional bifurcation problem

$$\nabla_x I(x,\lambda) = 0 \tag{4}$$
$$\text{where } I : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \quad I(x,\lambda) := \frac{1}{2} x^\top A x - \lambda \sin(|x|^p)$$

for some $p \in (1,2)$ and a symmetric, positive definite matrix $A \in \mathbb{R}^{n\times n}$.

**(a)** Show that, for $\lambda \in \mathbb{R}$, $I(\,\cdot\,,\lambda)$ is differentiable with respect to $x$ and calculate $\nabla_x I(\,\cdot\,,\lambda)$. Verify that $\nabla_x I(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$.

**(b)** Prove that, for $\lambda \in \mathbb{R}$, $I(\,\cdot\,,\lambda)$ admits a minimizer $x_\lambda \in \mathbb{R}^n$.

**(c)** Prove that $(0,0)$ is a bifurcation point for problem (4).

## Solution

**(a)** We fix $\lambda \in \mathbb{R}$ and $i \in \{1,...,n\}$. For $x \in \mathbb{R}^n$, we have

$$I(x,\lambda) = \frac{1}{2}\sum_{j,k=1}^n a_{jk} x_j x_k - \lambda \sin(|x|^p).$$

For $x \neq 0$, the chain rule gives partial differentiability and, using the symmetry of $A$,

$$\begin{aligned} I_{x_i}(x,\lambda) &= \frac{1}{2}\sum_{j,k=1}^n a_{jk}(\delta_{ij}x_k + x_j\delta_{ik}) - \lambda\cos(|x|^p)\cdot p x_i |x|^{p-2} \\ &= \sum_{j}^n a_{ij}x_j - \lambda\cos(|x|^p)\cdot p x_i|x|^{p-2} = \Big[Ax - \lambda\cos(|x|^p)\cdot p|x|^{p-2}x\Big]_i. \end{aligned}$$

For $x = 0$, we compute directly

$$I_{x_i}(0,\lambda) = \lim_{h\to 0}\frac{I(he_i,\lambda) - I(0,\lambda)}{h} = \lim_{h\to 0}\frac{\frac{1}{2}a_{ii}h^2 - \lambda\sin(|h|^p)}{h} = 0, \quad \text{and}$$
$$\lim_{h\to 0} I_{x_i}(h,\lambda) = 0$$

since $p > 1$. We conclude that $I$ is indeed continuously differentiable with

$$\nabla_x I(x,\lambda) = \begin{cases} Ax - \lambda\cos(|x|^p)\cdot p|x|^{p-2}x, & \lambda\in\mathbb{R}, x\neq 0, \\ 0, & \lambda \in \mathbb{R}, x = 0 \end{cases}$$

and in particular, we have the trivial family of solutions $\nabla_x I(0,\lambda) = 0$, $\lambda \in \mathbb{R}$. □

**(b)** For $x \in \mathbb{R}^n$, $\lambda > 0$ we estimate

$$I(x,\lambda) \geq \frac{\alpha}{2}|x|^2 - \lambda \sin(|x|^p) \geq -\lambda, \qquad (\heartsuit)$$

hence $j_\lambda := \inf_{x\in\mathbb{R}^n} I(x,\lambda) \geq -\lambda$ and we find a minimizing sequence $(x_\lambda^{(n)})_{n\in\mathbb{N}}$, i.e.

$$\lim_{n\to\infty} I(x_\lambda^{(n)},\lambda) = j_\lambda.$$

Taking another look at the first estimate in $(\heartsuit)$, we see that $(x_\lambda^{(n)})_{n\in\mathbb{N}}$ is necessarily bounded and hence, working in finite dimension, the Bolzano-Weierstrass theorem asserts that we can assume without loss of generality (possibly after passing to a subsequence) that $(x_\lambda^{(n)})_{n\in\mathbb{N}}$ converges to some element $x_\lambda \in \mathbb{R}^n$.

It then follows from the continuity of $I$ that $j_\lambda = \lim_{n\to\infty} I(x_\lambda^{(n)},\lambda) = I(x_\lambda,\lambda)$, and thus $x_\lambda$ is a minimizer for $I(\,\cdot\,,\lambda)$. □

**(c)** To find non-trivial solutions of (4), we consider the minimizers $x_\lambda$ of $I(\,\cdot\,,\lambda)$, which are stationary points of $I(\,\cdot\,,\lambda)$ and hence provide solutions of (4). We will prove the following:

*Assertion 1*: For every $\lambda > 0$, $x_\lambda \neq 0$.

*Assertion 2*: We have $x_\lambda \to 0$ as $\lambda \searrow 0$.

This finally yields that $(0,0)$ is a bifurcation point of problem (4) since it is a point where the non-trivial family $\{(x_\lambda,\lambda) : \ \lambda > 0\}$ meets the trivial branch of solutions.

We now prove the assertions 1 and 2. First of all, since $A$ is positive definite, we find $\beta \geq \alpha > 0$ with

$$\alpha|x|^2 \leq x^\top A x \leq \beta|x|^2 \qquad \text{for } x \in \mathbb{R}^n. \qquad (\spadesuit)$$

*Proof of Assertion 1*:
If $x_\lambda = 0$ held true, then we would find $j_\lambda = 0$ and therefore $I(\,\cdot\,,\lambda) \geq 0$. We aim to show, however, that $I(\,\cdot\,,\lambda)$ attains negative values. To this end, we recall that

$$\sin(\xi) \geq \frac{2}{\pi}\xi \qquad \text{for } \xi \in \left[0, \frac{\pi}{2}\right].$$

For $x \in \mathbb{R}^n$ with $|x|^p \leq \frac{\pi}{2}$, we estimate

$$I(x,\lambda) \leq \frac{\beta}{2}|x|^2 - \frac{2\lambda}{\pi}|x|^p = \frac{\beta}{2}|x|^p\left(|x|^{2-p} - \frac{4\lambda}{\beta\pi}\right).$$

Hence, choosing $x \in \mathbb{R}^n$ with $0 < |x| < \min\left\{\left(\frac{\pi}{2}\right)^{\frac{1}{p}}, \left(\frac{4\lambda}{\beta\pi}\right)^{\frac{1}{2-p}}\right\}$, we have $I(x,\lambda) < 0$, and the assertion is proved. Note that such a choice is possible because $p < 2$.

*Proof of Assertion 2*:
Since $x_\lambda \neq 0$, we have for all $\lambda > 0$

$$\nabla_x I(x_\lambda, \lambda) \quad \Leftrightarrow \quad Ax_\lambda = \lambda \cos(|x_\lambda|^p) \cdot p|x_\lambda|^{p-2} x_\lambda.$$

Multiplying from the left with $x_\lambda^\top$, we find in view of (♠):

$$(0 <) \;\; \alpha \leq \lambda \cos(|x_\lambda|^p) \cdot p|x_\lambda|^{p-2} \leq \beta.$$

Letting $\lambda \searrow 0$, the estimate from below can only hold if $\cos(|x_\lambda|^p) \cdot |x_\lambda|^{p-2} \to \infty$. As the cosine is bounded and $p < 2$, we infer $|x_\lambda| \to 0$ as $\lambda \searrow 0$. ☐