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Bifurcation Theory

Problem Sheet 3

Problem 8 (Remarks on Theorem II.3)

(a) Let $\alpha_0 > 0$ and assume that $G \in C^1((0,\alpha_0),\mathbb{R})$ with $G'(\alpha) > 0$ for $0 < \alpha < \alpha_0$. Prove that, for every $\alpha \in (0,\alpha_0)$, the following integrability condition holds:

$$\int_0^\alpha \frac{1}{\sqrt{G(\alpha)-G(z)}}\,\mathrm{d}z < \infty.$$

(b) Consider the problem

$$\begin{cases} -u'' = g(u,\lambda) & \text{in } (0,T), \\ u(0) = u(T) = 0 \end{cases} \tag{1}$$

with $g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$ as in Theorem II.3. Assume additionally that there exists $g_0 \in C(\mathbb{R},\mathbb{R})$ with

$$g(z,\lambda) = \lambda \cdot g_0(z) \qquad (z \in \mathbb{R}, \lambda \in \mathbb{R}).$$

Prove the following (stronger) version of Theorem II.3:

(i) For every $j \in \mathbb{N}_0$ and $\alpha \in (0,\alpha_0)$, there exists a j-nodal solution of (1) with $\|\cdot\|_\infty$-norm α.

(ii) If $g_0 \in C^1(\mathbb{R},\mathbb{R})$ and $j \in \mathbb{N}_0$, j-nodal solutions of (1) bifurcate from $(0,\lambda_j)$ in $C([0,T])$ if and only if

$$\lambda_j \cdot g_0'(0) = \left((j+1)\frac{\pi}{T}\right)^2.$$

Problem 9 (The direction of bifurcation)

Let $T > 0$ and, for $\lambda > 0$ and $\omega \in C([0,T],\mathbb{R})$, $\omega \geq 0$, consider the boundary value problem

$$\begin{cases} -u'' = \lambda\,\omega(x)\sin(u) & \text{in } (0,T), \\ u(0) = u(T) = 0. \end{cases} \tag{2}$$

Find $\lambda_j \in (0,\infty)$, $j \in \mathbb{N}_0$, with $\lambda_j \nearrow \infty$ and with the following property: If $u \in C^2([0,T])$ is a j-nodal solution of (2), then necessarily $\lambda \geq \lambda_j$.

Hint: Use the Sturm Comparison Theorem as introduced in the problem class.

Problem 10 (Gâteaux Differentiability)

Let X and Z be Banach spaces, $U \subseteq X$ an open subset and $x \in U$. A map $F : U \to Z$ is said to be *Gâteaux differentiable* in x if there exists a continuous linear operator $A \in \mathcal{L}(X,Z)$ with the property that, for every $h \in X$,

$$\lim_{\tau \to 0} \frac{F(x+\tau h) - F(x)}{\tau} = Ah.$$

In this case, we call $A =: \mathrm{d}F(x)$ the *Gâteaux derivative* of F in x.

(a) Assume that $F : U \to Z$ is Gâteaux differentiable in every $x \in U$, and that the map

$$\mathrm{d}F : U \to \mathcal{L}(X,Z), \quad x \mapsto \mathrm{d}F(x)$$

is continuous. Prove that F is continuously Fréchet differentiable on U, and that its Fréchet derivative is given by the Gâteaux derivative, i.e. $F'(x)[h] = \mathrm{d}F(x)[h]$ holds for all $x \in U, h \in X$.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\varphi \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. We consider the Banach space $C(\overline{\Omega})$ endowed with the norm $\|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|$, $u \in C(\overline{\Omega})$.

Prove that the map

$$F : C(\overline{\Omega},\mathbb{R}) \to C(\overline{\Omega},\mathbb{R}), \quad (F(u))(x) := \varphi(x, u(x)) \qquad (x \in \overline{\Omega})$$

is continuously Fréchet differentiable and calculate its derivative.