Problem 8 (Remarks on Theorem II.3)

(a) Let $\alpha_0 > 0$ and assume that $G \in C^1((0, \alpha_0), \mathbb{R})$ with $G'(\alpha) > 0$ for $0 < \alpha < \alpha_0$. Prove that, for every $\alpha \in (0, \alpha_0)$, the following integrability condition holds:

$$
\int_0^\alpha \frac{1}{\sqrt{G(\alpha) - G(z)}} \, dz < \infty.
$$

(b) Consider the problem

$$
\begin{cases}
-u'' = g(u, \lambda) & \text{in } (0, T), \\
u(0) = u(T) = 0
\end{cases}
$$

with $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ as in Theorem II.3. Assume additionally that there exists $g_0 \in C(\mathbb{R}, \mathbb{R})$ with

$$g(z, \lambda) = \lambda \cdot g_0(z) \quad (z \in \mathbb{R}, \lambda \in \mathbb{R}).$$

Prove the following (stronger) version of Theorem II.3:

(i) For every $j \in \mathbb{N}_0$ and $\alpha \in (0, \alpha_0)$, there exists a $j$-nodal solution of (1) with $\| \cdot \|_\infty$-norm $\alpha$.

(ii) If $g_0 \in C^1(\mathbb{R}, \mathbb{R})$ and $j \in \mathbb{N}_0$, $j$-nodal solutions of (1) bifurcate from $(0, \lambda_j)$ in $C([0, T])$ if and only if

$$
\lambda_j \cdot g_0'(0) = \left( (j + 1) \frac{\pi}{T} \right)^2.
$$
Solution

(a) Let $\alpha \in (0, \alpha_0)$. By continuity of $G'$ on $(0, \alpha]$, we find $\alpha_1 \in (0, \alpha)$ with

$$G'(^\prime) > \frac{G'(\alpha)}{\alpha} > 0 \quad \text{for } \alpha_1 < \xi < \alpha,$$

and the mean value theorem yields $\xi \in (z, \alpha)$ with $G(\alpha) - G(z) = G'(^\prime)(\alpha - z)$, hence

$$G(\alpha) - G(z) = G'(^\prime)(\alpha - z) > \frac{G'(\alpha)}{\alpha} (\alpha - z) \quad \text{for } \alpha_1 < z < \alpha. \quad (\Diamond)$$

Furthermore, $G' > 0$ on $(0, \alpha_0)$ implies that $G$ is strictly increasing, hence in particular

$$G(\alpha) - G(z) > G(\alpha) - G(\alpha_1) > 0 \quad \text{for } 0 < z < \alpha_1. \quad (\bigtriangledown)$$

We now estimate the improper Riemann integral

$$\int_0^\alpha \frac{1}{\sqrt{G(\alpha) - G(z)}} \, dz = \int_0^{\alpha_1} \frac{1}{\sqrt{G(\alpha) - G(z)}} \, dz + \int_{\alpha_1}^\alpha \frac{1}{\sqrt{G(\alpha) - G(z)}} \, dz$$

$$\leq \int_0^{\alpha_1} \frac{1}{\sqrt{G(\alpha) - G(\alpha_1)}} \, dz + \int_{\alpha_1}^\alpha \frac{1}{\sqrt{G'(\alpha)}} \, dz$$

$$= \frac{\alpha_1}{\sqrt{G(\alpha) - G(\alpha_1)}} + \sqrt{\frac{2}{G'(\alpha)}} \cdot 2\sqrt{\alpha - \alpha_1} < \infty.$$

This closes the proof. \hfill \Box

Remark:
This result shows that the integrability condition of Theorem II.3 is always satisfied.

(b) We let $g_0 \in C(\mathbb{R}, \mathbb{R})$, $T > 0$, $\alpha_0 > 0$ and $j \in \mathbb{N}_0$. Moreover, $G_0(z) := \int_0^z g_0(s) \, ds$ for $z \in \mathbb{R}$. We aim to prove the following stronger version of Theorem II.3:

**Corollary.** Assume that $g_0(-z) = -g_0(z)$ and $g_0(z) > 0$ hold for for $0 < |z| \leq \alpha_0$, and consider $\lambda > 0$. Then we have:

(i) For all $\alpha \in (0, \alpha_0)$ the boundary value problem

$$\begin{cases}
-u'' = \lambda g_0(u) & \text{in } (0, T), \\
u(0) = u(T) = 0
\end{cases}$$

admits, for some $\lambda = \lambda_j(\alpha) > 0$, a periodic $j$-nodal solution $u \in C^2([0, T])$ with periodicity $\frac{2T}{j+1}$ and which is semi-explicitly given by

$$\sqrt{2\lambda} \cdot x = \int_0^u \frac{1}{\sqrt{G_0(\alpha) - G_0(z)}} \, dz, \quad 0 \leq x \leq \frac{T}{2(j+1)}.$$

(ii) If $g_0 \in C^1(\mathbb{R}, \mathbb{R})$, $j$-nodal solutions bifurcate from $(0, \lambda_j)$ in $C([0, T])$ if and only if

$$\lambda_j \cdot g_0'(0) = \left( (j + 1) \pi / 2T \right)^2.$$
**Proof.** As $g_0$ is continuous, we have that $G_0 \in C^1(\mathbb{R}, \mathbb{R})$ with $G_0'(\alpha) = g_0(\alpha) > 0$ for $0 < \alpha < \alpha_0$ by assumption on $g_0$. Hence, part (a) yields that
\[
\int_0^\alpha \frac{1}{\sqrt{G_0'(\alpha) - G_0'(z)}} \, dz < \infty \quad \text{for all } \alpha \in (0, \alpha_0).
\]

We now apply Theorem II.3 with $g(z, \lambda) = \lambda \cdot g_0(z)$ and $G(z, \lambda) = \lambda \cdot G_0(z)$ for $\lambda > 0$. It states that a $j$-nodal solution with norm $\alpha \in (0, \alpha_0)$ exists under the condition
\[
\frac{T}{j + 1} = \int_0^\alpha \frac{1}{\sqrt{\frac{1}{2}(G(\alpha, \lambda) - G(z, \lambda))}} \, dz = \sqrt{\frac{2}{\lambda}} \int_0^\alpha \frac{1}{\sqrt{G_0(\alpha) - G_0(z)}} \, dz,
\]
which can be satisfied by choosing
\[
\lambda = \lambda_j(\alpha) := 2 \left( \frac{j + 1}{T} \int_0^\alpha \frac{1}{\sqrt{G_0(\alpha) - G_0(z)}} \, dz \right)^2.
\]
The semi-explicit formula is another direct consequence of Theorem II.3 as presented in the lecture.

Now, let $g_0$ (and hence $g$) be continuously differentiable. If bifurcation of $j$-nodal solutions occurs at $(0, \lambda_j)$ for some $\lambda_j > 0$, Theorem II.3 implies
\[
\left( (j + 1) \frac{\pi}{T} \right)^2 = g_0(0) = \lambda_j \cdot g_0'(0).
\]
Conversely, if that identity holds, then in particular $g_0'(0) > 0$, hence $g_0(0, \lambda_j) = g_0'(0) \neq 0$, and Theorem II.3 states that bifurcation of $j$-nodal solutions occurs at $(0, \lambda_j)$.

\[ \square \]
Problem 9 (The direction of bifurcation)

Let $T > 0$ and, for $\lambda > 0$ and $\omega \in C([0, T], \mathbb{R})$, $\omega \geq 0$, consider the boundary value problem

\[
\begin{aligned}
-u'' &= \lambda \omega(x) \sin(u) \quad \text{in } (0, T), \\
u(0) &= u(T) = 0.
\end{aligned}
\]  

(2)

Find $\lambda_j \in (0, \infty)$, $j \in \mathbb{N}_0$, with $\lambda_j \nearrow \infty$ and with the following property: If $u \in C^2([0, T])$ is a $j$-nodal solution of (2), then necessarily $\lambda \geq \lambda_j$.

Hint: Use the Sturm Comparison Theorem as introduced in the problem class.

Solution

First, let us note that for $\omega \equiv 0$, problem (2) admits only the trivial solution. We hence assume $\omega \not\equiv 0$.

Let $(\lambda, u) \in C^2([0, T]) \times (0, \infty)$ be a solution of problem (2) where $u$ is $j$-nodal. We thus find an interval $(\alpha, \beta) \subseteq (0, T)$ with the following properties:

$u(\alpha) = u(\beta) = 0$, $u(x) \neq 0$ for all $x \in (\alpha, \beta)$, $0 < \beta - \alpha \leq \frac{T}{j + 1}$.

We now define $\tilde{u} : \mathbb{R} \to \mathbb{R}$, $\tilde{u}(x) := \sin \left( \frac{\pi}{\beta - \alpha} x - \alpha \right)$ and let $\tilde{\lambda} := \left( \frac{\pi}{\beta - \alpha} \right)^2$ and have

\[
\begin{aligned}
u'' + \lambda \omega(x) \psi(u) \ u &= 0 \quad \text{on } (\alpha, \beta), \\
u(\alpha) &= 0, \ u(\beta) = 0,
\end{aligned}
\]

\[
\begin{aligned}
\tilde{u}'' + \tilde{\lambda} \tilde{u} &= 0 \quad \text{on } (\alpha, \beta), \\
\tilde{u}(\alpha) &= 0, \ \tilde{u}(\beta) = 0.
\end{aligned}
\]

(♠)

with the smooth function $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(z) := \begin{cases} \sin(z) & z \neq 0, \\ 1 & z = 0, \end{cases}$ satisfying $\psi \leq 1$ on $\mathbb{R}$.

Assume for contradiction that $\lambda < \lambda_j := \frac{1}{\|\omega\|_{\infty}} \cdot \left( \frac{\pi(j+1)}{T} \right)^2$. This implies for all $x \in (\alpha, \beta)$

$\lambda \omega(x) \psi(u(x)) \leq \lambda \|\omega\|_{\infty} < \lambda_j \|\omega\|_{\infty} = \left( \frac{\pi(j+1)}{T} \right)^2 \leq \left( \frac{\pi}{\beta - \alpha} \right)^2 = \tilde{\lambda}$.

We recall that $\alpha$ and $\beta$ are successive zeros of both $u$ and $\tilde{u}$. Hence, by the Sturm Comparison Theorem applied to (♠), there exists $x_0 \in (\alpha, \beta)$ with $\tilde{u}(x_0) = 0$, which is the contradiction we were heading for.

We conclude that

$\lambda \geq \lambda_j$ where $\lambda_j = \frac{1}{\|\omega\|_{\infty}} \cdot \left( \frac{\pi(j+1)}{T} \right)^2$,

and remark that for $\omega \equiv 1$ (in the case of the pendulum equation), these $\lambda_j$ coincide with the bifurcation points we found in the lecture, i.e. that bifurcation directs “to the right“.
Problem 10 (Gâteaux Differentiability)

Let $X$ and $Z$ be Banach spaces, $U \subseteq X$ an open subset and $x \in U$. A map $F : U \to Z$ is said to be \textit{Gâteaux differentiable} in $x$ if there exists a continuous linear operator $A \in \mathcal{L}(X, Z)$ with the property that, for every $h \in X$,

$$
\lim_{\tau \to 0} \frac{F(x + \tau h) - F(x)}{\tau} = Ah.
$$

In this case, we call $A =: dF(x)$ the \textit{Gâteaux derivative} of $F$ in $x$.

(a) Assume that $F : U \to Z$ is Gâteaux differentiable in every $x \in U$, and that the map

$$
dF : U \to \mathcal{L}(X, Z), \quad x \mapsto dF(x)
$$

is continuous. Prove that $F$ is continuously Fréchet differentiable on $U$, and that its Fréchet derivative is given by the Gâteaux derivative, i.e. $F'(x)[h] = dF(x)[h]$ holds for all $x \in U, h \in X$.

(b) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $\varphi \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. We consider the Banach space $C(\overline{\Omega})$ endowed with the norm $\|u\|_\infty := \max_{x \in \Omega} |u(x)|$, $u \in C(\overline{\Omega})$.

Prove that the map

$$
F : C(\overline{\Omega}, \mathbb{R}) \to C(\overline{\Omega}, \mathbb{R}), \quad (F(u))(x) := \varphi(x, u(x)) \quad (x \in \overline{\Omega})
$$

is continuously Fréchet differentiable and calculate its derivative.

Solution

(a) Let $x \in U$ and $\varepsilon > 0$. By continuity of the Gâteaux derivative, we find $\delta > 0$ with $B_\delta(x) \subseteq U$ and

$$
\|dF(x + y) - dF(x)\|_{\mathcal{L}(X, Z)} < \varepsilon \quad \text{whenever} \quad y \in X, \|y\|_X < \delta. \quad (\clubsuit)
$$

We consider an arbitrary element of the dual space $z' \in Z'$ and $h \in X$ with $\|h\|_X < \delta$ and (without loss of generality) so small that the following function is well-defined:

$$
f : (-1, 1) \to \mathbb{R}, \quad f(\tau) := z'(F(x + \tau h)).
$$

\textbf{Assertion:} $f$ is differentiable with $f'(\tau) = z'(dF(x + \tau h)[h])$.

\textbf{Proof:} For $\sigma \in \mathbb{R}$, $\sigma \to 0$, we have with $\tilde{x} := x + \tau h$

$$
|f(\tau + \sigma) - f(\tau) - \sigma z'(dF(x + \tau h)[h])|
= |z'(F(x + \tau h + \sigma h) - F(x + \tau h) - \sigma dF(x + \tau h)[h])|
\leq \|z'\|_{Z'} \|F(x + \tau h + \sigma h) - F(x + \tau h) - \sigma dF(x + \tau h)[h]\|_Z
= \|z'\|_{Z'} \|F(\tilde{x} + \sigma h) - F(\tilde{x}) - \sigma dF(\tilde{x})[h]\|_Z = o(\sigma)
$$
As a consequence, the fundamental theorem of calculus gives
\[
z'(F(x + h) - F(x) - dF(x)[h]) = f(1) - f(0) - f'(0) = \int_0^1 f'(\tau) - f'(0) \, d\tau
= \int_0^1 z'(dF(x + \tau h)[h] - dF(x)[h]) \, d\tau.
\]
Since \(\|\tau h\|_X \leq \|h\|_X < \delta\) for \(0 \leq \tau \leq 1\), estimate (♣) yields
\[
|z'(F(x + h) - F(x) - dF(x)[h])| \leq \int_0^1 ||z'||_Z \|dF(x + \tau h) - dF(x)\|_L(X,Z) \|h\|_X \, d\tau
\leq \varepsilon \|z'||_Z \|h\|_X.
\]
By a consequence of the Hahn-Banach Theorem\(^1\), we may conclude
\[
\|F(x + h) - F(x) - dF(x)h\|_Z \leq \varepsilon \|h\|_X
\]
for all \(h \in X\) with \(\|h\|_X \leq \delta\) since \(z' \in Z'\) was arbitrary. This yields \(F(x + h) - F(x) - dF(x)[h] = o(\|h\|_X)\); hence, \(F\) is Fréchet differentiable in every \(x \in U\) with \(F'(x) = dF(x)\), and continuity of the Fréchet derivative is a direct consequence of continuity of the Gâteaux derivative (which was assumed). \(\square\)

(b) We intend to apply the result of part (a); this requires two steps.\(^2\)

**Step 1: Gâteaux differentiability of** \(F\)

Let \(u, h \in C(\overline{\Omega})\). By definition of Gâteaux differentiability, we have to find a linear continuous operator \(A \in L(C(\overline{\Omega}), C(\overline{\Omega}))\) with
\[
\lim_{\tau \to 0} \frac{\|F(u + \tau h) - F(u) - \tau Ah\|_\infty}{\tau} = 0.
\]
First, we prove the existence of a pointwise limit and thus fix \(x \in \overline{\Omega}\). Then,
\[
\lim_{\tau \to 0} \frac{(F(u + \tau h))(x) - (F(u))(x)}{\tau} = \lim_{\tau \to 0} \frac{\varphi(x, u(x) + \tau h(x)) - \varphi(x, u(x))}{\tau}
= \frac{\partial \varphi}{\partial x_{n+1}}(x, u(x)) \cdot h(x) =: \varphi_u(x, u(x)) \cdot h(x)
\]
which is a consequence of the chain rule and of the differentiability of \(\varphi\). Hence, the only candidate for a Gâteaux derivative is \((dF(u)[h])(x) = \varphi_u(x, u(x)) \cdot h(x)\).

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\(^1\)for a reference in German, cf. Werner, Funktionalanalysis (7. Auflage), Korollar III.1.6; we use that, for \(z \in Z\), \(\|z\|_Z = \sup\{z'(z) : z' \in Z', ||z'||_{Z'} = 1\}\).

\(^2\)cf. Chang, Methods in Nonlinear Analysis, p. 4
In a second step, we have to show that this limit is uniform (i.e. that we have convergence in the Banach space $C(\overline{\Omega})$). Let $\varepsilon > 0$. Since $\varphi_u$ is uniformly continuous on the compact set $\overline{\Omega} \times [-M, M]$, $M := \|u\|_{\infty} + \|h\|_{\infty}$, we find $\delta > 0$ with

$$\forall x \in \overline{\Omega}, \forall z_1, z_2 \in [-M, M]: \quad |z_1 - z_2| < \delta \Rightarrow |\varphi_u(x, z_1) - \varphi_u(x, z_2)| < \frac{\varepsilon}{\max\{1, \|h\|_{\infty}\}}.$$ 

So for all $\tilde{\tau} \in [-1, 1]$ with $|\tilde{\tau}| \|h\|_{\infty} < \delta$, we have

$$\forall x \in \overline{\Omega}: \quad |\varphi_u(x, u(x) + \tilde{\tau} h(x)) - \varphi_u(x, u(x))| < \frac{\varepsilon}{\max\{1, \|h\|_{\infty}\}}$$

and find for $|\tau| \in (0, 1)$, using the fundamental theorem of calculus,

$$\max_{x \in \overline{\Omega}} \left| \frac{\varphi(x, u(x) + \tau h(x)) - \varphi(x, u(x)) - \tau \varphi_u(x, u(x))h(x)}{\tau} \right|$$

$$= \max_{x \in \overline{\Omega}} \left| \frac{1}{\tau} \left( \int_0^1 \frac{d}{d\sigma} \left[ \varphi(x, u(x) + \sigma \tau h(x)) \right] d\sigma \right) \right|$$

$$= \max_{x \in \overline{\Omega}} \left| \int_0^1 \varphi_u(x, u(x) + \sigma \tau h(x))h(x) - \varphi_u(x, u(x))h(x) d\sigma \right|$$

$$\leq \max_{x \in \overline{\Omega}} \int_0^1 |\varphi_u(x, u(x) + \sigma \tau h(x)) - \varphi_u(x, u(x))| |h(x)| d\sigma$$

$$< \max_{x \in \overline{\Omega}} \int_0^1 \frac{\varepsilon}{\max\{1, \|h\|_{\infty}\}} \|h\|_{\infty} d\sigma = \frac{\varepsilon}{\max\{1, \|h\|_{\infty}\}} \|h\|_{\infty} \leq \varepsilon.$$ 

As the partial derivative $\varphi_u$ is continuous, we define

$$A : C(\overline{\Omega}) \to C(\overline{\Omega}), \quad (Ah)(x) := \varphi_u(x, u(x)) \cdot h(x), \quad x \in \Omega$$

the estimate above shows $\lim_{\tau \to 0} \frac{\|F(u+\tau h) - F(u) - \tau Ah\|_{\infty}}{\tau} = 0$. Furthermore, for fixed $u \in C(\overline{\Omega})$, $A$ is linear and continuous, and we conclude that $F$ is Gâteaux differentiable with derivative $dF(u) = A$.

**Step 2: Continuity of the Gâteaux derivative of $F$**

We consider functions $u_n, u \in C(\overline{\Omega})$, $n \in \mathbb{N}$, with $\|u_n - u\|_{\infty} \to 0$ as $n \to \infty$. We intend to prove that $dF(u_n) \to dF(u)$ as $n \to \infty$ in $L(C(\overline{\Omega}), C(\overline{\Omega}))$, i.e.

$$\sup_{h \in C(\overline{\Omega}), \|h\|_{\infty} = 1} \|dF(u_n)[h] - dF(u)[h]\|_{\infty} \to 0.$$ 

We let $\varepsilon > 0$ and, using uniform continuity of $\varphi_u$ in a similar way as in the previous part, we find $\delta > 0$ with

$$\forall x \in \overline{\Omega}, \forall z_1, z_2 \in [-M_1, M_1]: \quad |z_1 - z_2| < \delta \Rightarrow |\varphi_u(x, z_1) - \varphi_u(x, z_2)| < \varepsilon$$
where $M_1 := \|u\|_\infty + 1$. As $u_n \to u$ uniformly on $\Omega$, we also find $n_0 \in \mathbb{N}$ with $\|u_n\|_\infty \leq M_1$ and $|u_n(x) - u(x)| < \delta$ for all $n \geq n_0$ and $x \in \Omega$. We estimate for $n \geq n_0$

$$\sup_{h \in C(\Omega), \|h\|_\infty = 1} \|dF(u_n)[h] - dF(u)[h]\|_\infty \leq \sup_{h \in C(\Omega), \|h\|_\infty = 1} \left( \max_{x \in \Omega} |\phi_u(x, u_n(x)) - \phi_u(x, u(x))| \cdot \|h\|_\infty \right)$$

$$= \max_{x \in \Omega} |\phi_u(x, u_n(x)) - \phi_u(x, u(x))| < \varepsilon,$$

which shows continuity of the Gâteaux derivative. By part (a), we conclude that $F$ is continuously Fréchet differentiable with $F'(u)[h] = dF(u)[h]$. □