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Bifurcation Theory

Problem Sheet 4

Problem 11 (Differentiability of an integral operator)

Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ an open subset, $2 \le p < \frac{2n}{n-2}$ for $n \ge 3$ and $2 \le p < \infty$ else. Prove that the following map is continuously Fréchet differentiable

$$F : H_0^1(\Omega) \to \mathbb{R}, \quad F(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x$$

with derivative $F'(u)[h] = \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x - \int_\Omega |u|^{p-2}uh \,\mathrm{d}x$ where $u, h \in H_0^1(\Omega)$.

Hint: By choice of p, the continuous Sobolev embedding $H_0^1(\Omega) \subseteq L^p(\Omega)$ holds, i.e. there exists $C > 0$ with the property that $\|u\|_{L^p(\Omega)} \le C \|u\|_{H_0^1(\Omega)}$ for every $u \in H_0^1(\Omega)$.

Problem 12 (Directional derivatives)

On the Banach space $L^\infty(\mathbb{R})$, we consider the map $F : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}),\ F(u) := |u|^{\frac{1}{2}}$. Let $u_0 := \mathbb{1}_{[0,1]} \in L^\infty(\mathbb{R})$. For $h \in L^\infty(\mathbb{R})$, prove that the directional derivative

$$\lim_{\tau \to 0} \frac{F(u_0 + \tau h) - F(u_0)}{\tau}$$

exists if and only if $h(x) = 0$ for almost all $x \in \mathbb{R} \setminus [0,1]$. Conclude that F is not Gâteaux differentiable in the point u_0.

Problem 13 (An application of the Implicit Function Theorem)

For $\lambda \in \mathbb{R}$, we consider the boundary value problem

$$\begin{cases} u'' - \sin(u) = \lambda \mathrm{e}^x & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Show that there exists such $\delta > 0$ that (1) admits a solution $u_\lambda \in C^2([0,1],\mathbb{R}) \setminus \{0\}$ for $0 < |\lambda| < \delta$.