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Bifurcation Theory

Solutions to Problem Sheet 4

Problem 11 (Differentiability of an integral operator)

Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ an open subset, $2 \leq p < \frac{2n}{n-2}$ for $n \geq 3$ and $2 \leq p < \infty$ else. Prove that the following map is continuously Fréchet differentiable

$$F : H_0^1(\Omega) \to \mathbb{R}, \quad F(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p} \int_\Omega |u|^p \,\mathrm{d}x$$

with derivative $F'(u)[h] = \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x - \int_\Omega |u|^{p-2} uh \,\mathrm{d}x$ where $u, h \in H_0^1(\Omega)$.

Hint: By choice of p, the continuous Sobolev embedding $H_0^1(\Omega) \subseteq L^p(\Omega)$ holds, i.e. there exists $C > 0$ with the property that $\|u\|_{L^p(\Omega)} \leq C \|u\|_{H_0^1(\Omega)}$ for every $u \in H_0^1(\Omega)$.

Solution

We discuss seperately $F_0, F_1 : H_0^1(\Omega) \to \mathbb{R}$,

$$F_0(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \qquad F_1(u) := \frac{1}{p} \int_\Omega |u|^p \,\mathrm{d}x.$$

Continuous Fréchet differentiability of F_0:

For $u, h \in H_0^1(\Omega)$, we calculate directly:

$$\begin{aligned} F_0(u+h) - F_0(u) &= \frac{1}{2} \int_\Omega \nabla(u+h) \cdot \nabla(u+h) - \nabla u \cdot \nabla u \,\mathrm{d}x \\ &= \int_\Omega \nabla u \cdot \nabla h + \frac{1}{2} |\nabla h|^2 \,\mathrm{d}x \end{aligned}$$

and hence, using the norm $\|h\|^2_{H^1(\Omega)} := \int_\Omega |\nabla h|^2 + h^2 \,\mathrm{d}x$,

$$\left| F_0(u+h) - F_0(u) - \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x \right| = \frac{1}{2} \int_\Omega |\nabla h|^2 \,\mathrm{d}x \leq \frac{1}{2} \|h\|^2_{H^1(\Omega)} = o(\|h\|_{H^1(\Omega)});$$

moreover, $H_0^1(\Omega) \to \mathbb{R},\ \ h \mapsto \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x$ is a continuous linear operator. We conclude that F_0 is Fréchet differentiable on $H_0^1(\Omega)$ with $F_0'(u)[h] = \int_\Omega \nabla u \cdot \nabla h \,\mathrm{d}x$ for $u, h \in H_0^1(\Omega)$. To see (even Lipschitz) continuity of F_0', we estimate for $u, v \in H_0^1(\Omega)$

$$\begin{aligned}\|F_0'(u) - F_0'(v)\|_{\mathcal{L}(H_0^1(\Omega),\mathbb{R})} &= \sup_{h \in H_0^1(\Omega),\, \|h\|_{H^1(\Omega)}=1} |F_0'(u)[h] - F_0'(v)[h]| \\ &\le \sup_{h \in H_0^1(\Omega),\, \|h\|_{H^1(\Omega)}=1} \int_\Omega |\nabla(u-v)\cdot\nabla h| \,\mathrm{d}x \\ &\le \sup_{h \in H_0^1(\Omega),\, \|h\|_{H^1(\Omega)}=1} \big(\|u-v\|_{H^1(\Omega)} \|h\|_{H^1(\Omega)} \big) = \|u-v\|_{H^1(\Omega)}.\end{aligned}$$

Continuous Fréchet differentiability of F_1:

We intend to apply the result of Problem 10 (a) and prove continuous Gâteaux differentiability.

Step 1: Gâteaux differentiability of F_1

Let $u, h \in H_0^1(\Omega)$. By definition of Gâteaux differentiability, we have to prove that

$$\mathbb{R} \to \mathbb{R}, \quad \tau \mapsto F_1(u+\tau h) = \frac{1}{p}\int_\Omega |u+\tau h|^p \,\mathrm{d}x$$

is differentiable at $\tau = 0$ with derivative $\int_\Omega |u|^{p-2} u h \,\mathrm{d}x$. This will be achieved via a standard consequence of dominated convergence (differentiation under the integral sign).

We let $f(\tau, x) := \frac{1}{p}|u(x) + \tau h(x)|^p$ for $x \in \Omega, \tau \in \mathbb{R}$. Then, for $x \in \Omega$, f is differentiable w.r.t. τ, and $\frac{\partial f}{\partial \tau}(\tau, x) = |u(x) + \tau h(x)|^{p-2}(u(x) + \tau h(x))h(x)$. Moreover, for $-1 \le \tau \le 1$, this expression is majorized by

$$\left|\frac{\partial f}{\partial \tau}(\tau, x)\right| \le |u(x) + \tau h(x)|^{p-1}|h(x)| \le (|u(x)| + |h(x)|)^p \le 2^p(|u(x)|^p + |h(x)|^p),$$

which is integrable due to the Sobolev embedding given in the hint. Dominated convergence now gives, for $-1 < \tau < 1$, existence of the derivative

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\big(F_1(u+\tau h)\big) = \frac{\mathrm{d}}{\mathrm{d}\tau}\int_\Omega f(\tau,x)\,\mathrm{d}x = \int_\Omega \frac{\partial f}{\partial \tau}(\tau,x)\,\mathrm{d}x = \int_\Omega |u+\tau h|^{p-2}(u+\tau h)h\,\mathrm{d}x.$$

Moreover, setting $\tau = 0$ and for fixed $u \in H_0^1(\Omega)$, the operator $H_0^1(\Omega) \to \mathbb{R},\ \ h \mapsto \int_\Omega |u|^{p-2} u h \,\mathrm{d}x$ is linear and continuous. (The latter is a consequence of the continuous Sobolev embedding in the hint.) Hence, we have Gâteaux differentiability of F_1 with Gâteaux derivative

$$\mathrm{d}F_1(u)[h] = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0}\big(F_1(u+\tau h)\big) = \int_\Omega |u|^{p-2} u h \,\mathrm{d}x.$$

Step 2: Continuity of the Gâteaux derivative of F_1

We let $u, v_n \in H_0^1(\Omega)$ with $\|v_n\|_{H_0^1(\Omega)} \to 0$ as $n \to \infty$. By the Sobolev embedding in the hint, we also have $\|v_n\|_{L^p(\Omega)} \to 0$ as $n \to \infty$.

The proof of the Riesz-Fischer Theorem shows that we may pass to a subsequence with $v_{n_k}(x) \to 0$ and $|v_{n_k}(x)| \le \varphi(x)$ for almost all $x \in \Omega$ and for some function $\varphi \in L^p(\Omega)$. We then have almost everywhere on Ω

$$\begin{aligned} \left| |u+v_{n_k}|^{p-2}(u+v_{n_k}) - |u|^{p-2}u \right|^{\frac{p}{p-1}} &\le \left((|u| + |\varphi|)^{p-1} + |u|^{p-1} \right)^{\frac{p}{p-1}} \\ &\le 2^{\frac{p}{p-1}} \cdot \left((|u| + |\varphi|)^p + |u|^p \right) \in L^1(\Omega), \end{aligned}$$

hence by dominated convergence

$$\int_\Omega \left| |u+v_{n_k}|^{p-2}(u+v_{n_k}) - |u|^{p-2}u \right|^{\frac{p}{p-1}} \mathrm{d}x \to 0 \qquad \text{as } k \to \infty.$$

The argument can be repeated when replacing $(v_n)_{n\in\mathbb{N}}$ by an arbitrary subsequence, yielding a sub-subsequence with the property shown above. Thus,

$$\int_\Omega \left| |u+v_n|^{p-2}(u+v_n) - |u|^{p-2}u \right|^{\frac{p}{p-1}} \mathrm{d}x \to 0 \qquad \text{as } n \to \infty.$$

Now, we can estimate the norms using Hölder's inequality

$$\begin{aligned} &\|\mathrm{d}F_1(u+v_n) - \mathrm{d}F_1(u)\|_{\mathcal{L}(H_0^1(\Omega),\mathbb{R})} = \sup_{h \in H_0^1(\Omega),\, \|h\|_{H^1(\Omega)}=1} |\mathrm{d}F_1(u+v_n)[h] - \mathrm{d}F_1(u)[h]| \\ &\le \sup_{h \in H_0^1(\Omega),\, \|h\|_{H^1(\Omega)}=1} \int_\Omega \left| |u+v_n|^{p-2}(u+v_n)h - |u|^{p-2}uh \right| \mathrm{d}x \\ &\le \left[\sup_{h \in H_0^1(\Omega),\, \|h\|_{H^1(\Omega)}=1} \|h\|_{L^p(\Omega)} \right] \cdot \left(\int_\Omega \left| |u+v_n|^{p-2}(u+v_n) - |u|^{p-2}u \right|^{\frac{p}{p-1}} \mathrm{d}x \right)^{\frac{p-1}{p}} \\ &\to 0 \qquad \text{as } n \to \infty \end{aligned}$$

since the supremum is finite due to the Sobolev embedding from the hint. This proves continuity, and together with Problem 10 (a) we have continuous Fréchet differentiability of F_1 and F. □

Problem 12 (Directional derivatives)

On the Banach space $L^\infty(\mathbb{R})$, we consider the map $F : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}),\ F(u) := |u|^{\frac{1}{2}}$. Let $u_0 := \mathbb{1}_{[0,1]} \in L^\infty(\mathbb{R})$. For $h \in L^\infty(\mathbb{R})$, prove that the directional derivative

$$\lim_{\tau\to 0} \frac{F(u_0 + \tau h) - F(u_0)}{\tau}$$

exists if and only if $h(x) = 0$ for almost all $x \in \mathbb{R} \setminus [0, 1]$. Conclude that F is not Gâteaux differentiable in the point u_0.

Solution

(i) First, we consider $h \in L^\infty(\mathbb{R})$ which does not satisfy $h(x) = 0$ a.e. on $\mathbb{R} \setminus [0, 1]$.

By assumption, there exists such $\delta > 0$ that the set $N := \{|h| \ge \delta\} \setminus [0, 1]$ has positive measure. Then for almost every $x \in N$, we have $u_0(x) = 0$ and $|h(x)| \ge \delta$ and estimate as follows for $\tau \neq 0$:

$$\left|\frac{(F(u_0 + \tau h))(x) - (F(u_0))(x)}{\tau}\right| = \left|\frac{|u_0(x) + \tau h(x)|^{\frac{1}{2}} - |u_0(x)|^{\frac{1}{2}}}{\tau}\right| = \left|\frac{h(x)}{\tau}\right|^{\frac{1}{2}} \ge \sqrt{\delta}|\tau|^{-\frac{1}{2}}.$$

Thus, as $\tau \to 0$,

$$\left\|\frac{F(u_0 + \tau h) - F(u_0)}{\tau}\right\|_{L^\infty(\mathbb{R})} \ge \left\|\frac{F(u_0 + \tau h) - F(u_0)}{\tau}\right\|_{L^\infty(N)} \ge \sqrt{\delta}|\tau|^{-\frac{1}{2}} \to \infty,$$

and hence the directional derivative in direction h does not exist.

(ii) Now, we let $h \in L^\infty(\mathbb{R})$ with $h(x) = 0$ a.e. on $\mathbb{R} \setminus [0, 1]$. We will prove that the directional derivative exists and is given by

$$\lim_{\tau\to 0} \frac{F(u_0 + \tau h) - F(u_0)}{\tau} = \frac{1}{2}h.$$

We only discuss the case $\|h\|_{L^\infty(\mathbb{R})} > 0$. For $\tau \in \mathbb{R}$ with $0 < |\tau| < \frac{1}{2\|h\|_{L^\infty(\mathbb{R})}}$, we have

$$\begin{aligned}\left\|\frac{F(u_0 + \tau h) - F(u_0)}{\tau} - \frac{1}{2}h\right\|_{L^\infty(\mathbb{R})} &= \left\|\frac{|u_0 + \tau h|^{\frac{1}{2}} - |u_0|^{\frac{1}{2}}}{\tau} - \frac{1}{2}h\right\|_{L^\infty([0,1])} \\ &= \operatorname*{ess\,sup}_{0\le x\le 1} \left|\frac{\sqrt{1 + \tau h(x)} - 1 - \frac{1}{2}\tau h(x)}{\tau}\right|.\end{aligned}$$

We estimate further by Taylor expansion. For almost every $x \in [0, 1]$, we have $|\tau h(x)| \le \frac{1}{2}$, and there exists $\xi(x, \tau) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ with

$$\sqrt{1 + \tau h(x)} = 1 + \tau h(x) \cdot \frac{1}{2} + \frac{1}{2}\tau^2 h(x)^2 \cdot \left(-\frac{1}{4}\right) (1 + \xi(x, \tau))^{-\frac{3}{2}},$$

hence we estimate the expression above

$$\begin{aligned}\left\| \frac{F(u_0 + \tau h) - F(u_0)}{\tau} - \frac{1}{2}h \right\|_{L^\infty(\mathbb{R})} &= \operatorname*{ess\,sup}_{0 \le x \le 1} \left| \frac{1}{\tau} \cdot \frac{1}{2}\tau^2 h(x)^2 \cdot \left(-\frac{1}{4}\right) (1 + \xi(x,\tau))^{-\frac{3}{2}} \right| \\ &\le \operatorname*{ess\,sup}_{0 \le x \le 1} \left| \frac{1}{8}\tau h(x)^2 \cdot (1 + \xi(x,\tau))^{-\frac{3}{2}} \right| \\ &\le \frac{1}{8}\tau \, \|h\|^2_{L^\infty(\mathbb{R})} \cdot \left(\frac{1}{2}\right)^{-\frac{3}{2}} = 2^{-\frac{3}{2}} \, \|h\|^2_{L^\infty(\mathbb{R})} \cdot \tau.\end{aligned}$$

We conclude that, as asserted,

$$\left\| \frac{F(u_0 + \tau h) - F(u_0)}{\tau} - \frac{1}{2}h \right\|_{L^\infty(\mathbb{R})} \to 0 \qquad \text{as} \quad \tau \to 0.$$

Finally, we infer that F is not Gâteaux differentiable at the point u_0. In fact, Gâteaux differentiability would in particular imply existence of all directional derivatives,

$$\lim_{\tau \to 0} \frac{F(u_0 + \tau h) - F(u_0)}{\tau} = \mathrm{d}F(u_0)[h],$$

thereby contradicting part (i). □

Problem 13 (An application of the Implicit Function Theorem)

For $\lambda \in \mathbb{R}$, we consider the boundary value problem

$$\begin{cases} u'' - \sin(u) = \lambda \mathrm{e}^x & \text{in } (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

Show that there exists such $\delta > 0$ that (1) admits a solution $u_\lambda \in C^2([0,1],\mathbb{R}) \setminus \{0\}$ for $0 < |\lambda| < \delta$.

Solution

We consider the Banach spaces $Z := C([0,1],\mathbb{R})$ with norm $\|u\|_{C([0,1])} := \max_{x\in[0,1]} |u(x)|$, $u \in Z$, and $X := \{u \in C^2([0,1],\mathbb{R}) : \ u(0) = u(1) = 0\}$ with norm $\|u\|_{C^2([0,1])} := \|u\|_{C([0,1])} + \|u'\|_{C([0,1])} + \|u''\|_{C([0,1])}$, $u \in X$.[1] Moreover, we define

$$F : X \times \mathbb{R} \to Z, \quad (F(u,\lambda))(x) := u''(x) - \sin(u(x)) - \lambda \mathrm{e}^x \quad (0 \le x \le 1).$$

Then every solution $u \in C^2([0,1],\mathbb{R})$ of (1) satisfies $u \in X$ and $F(u,\lambda) = 0$. Conversely, $u \in X$ and $F(u,\lambda) = 0$ implies that u is a C^2-solution of (1).

For $\lambda \in \mathbb{R}$, let us note that

$$F(0,\lambda) = 0 \quad \Leftrightarrow \quad \lambda = 0. \qquad (\diamondsuit)$$

In order to apply the Implicit Function Theorem in a neighborhood of the zero $(0,0)$ of F, we check differentiability of F.

Assertion: F is Fréchet differentiable w.r.t u, and for $u, h \in X$ and $\lambda \in \mathbb{R}$ the partial (Fréchet) derivative is $D_u F(u,\lambda)[h] = h'' - \cos(u) \cdot h$.

Proof:

For fixed $u \in X$ and $\lambda \in \mathbb{R}$, we let $Ah := h'' - \cos(u) \cdot h$ ($h \in X$). As the cosine is bounded, we have

$$A \in \mathcal{L}(X,Z) \qquad \text{with} \quad \|Ah\|_{C([0,1])} \le \|h\|_{C^2([0,1])} \quad (h \in X).$$

To prove Fréchet differentiability, we estimate for $h \in X$, twice using the fundamental

[1] X is complete because it is a closed subset of the Banach space $C^2([0,1],\mathbb{R})$ with norm $\|\cdot\|_{C^2([0,1])}$.

theorem of calculus,

$$\|F(u+h,\lambda)-F(u,\lambda)-Ah\|_{C([0,1])} = \max_{0\le x\le 1} |\sin((u+h)(x)) - \sin(u(x)) - \cos(u(x))\cdot h(x)|$$
$$= \max_{0\le x\le 1}\left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}\sin(u(x)+\tau h(x))\,\mathrm{d}\tau - \cos(u(x))\cdot h(x)\right|$$
$$= \max_{0\le x\le 1}\left|\int_0^1 \Big[\cos(u(x)+\tau h(x)) - \cos(u(x))\Big] h(x)\,\mathrm{d}\tau\right|$$
$$= \max_{0\le x\le 1}\left|\int_0^1 \Big[\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\sigma}\cos(u(x)+\sigma\tau h(x))\,\mathrm{d}\sigma\Big] h(x)\,\mathrm{d}\tau\right|$$
$$= \max_{0\le x\le 1}\left|\int_0^1 \Big[\int_0^1 \sin(u(x)+\sigma\tau h(x))\,\mathrm{d}\sigma\Big] \tau(h(x))^2\,\mathrm{d}\tau\right|$$
$$\le \int_0^1 \tau\,\|h\|^2_{C([0,1])}\,\mathrm{d}\tau = \frac{1}{2}\|h\|^2_{C([0,1])} \le \frac{1}{2}\|h\|^2_{C^2([0,1])} = o(\|h\|_{C^2([0,1])}).$$

■

We are now in a position to verify conditions (i) and (ii) of Theorem III.7 (IFT).

(i) We intend to show that the linear continuous operator $D_uF(0,0) : X \to Z,\ h \mapsto h'' - h$ is a homeomorphism.

It is sufficient to prove[2] that $D_uF(0,0)$ is injective, or equivalently, that the homogeneous boundary value problem

$$\begin{cases} h'' - h = 0 & \text{in}(0,1), \\ h(0) = h(1) = 0 \end{cases}$$

has only got the trivial solution. (Note that the boundary conditions are hidden in the definition of the space X.) To do this, we first write down the general solution of the differential equation

$$h(x) = \alpha\cosh(x) + \beta\sinh(x) \quad (0 \le x \le 1),$$

and calculate the parameters $\alpha, \beta \in \mathbb{R}$ from the boundary conditions:

$$0 = h(0) = \alpha, \qquad 0 = h(1) = \alpha\cosh(1) + \beta\sinh(1),$$

hence $\alpha = \beta = 0$ and $h \equiv 0$. The theorem cited in the footnote then states that $D_uF(0,0)$ is invertible. The Bounded Inverse Theorem[3] implies that $(D_uF(0,0))^{-1} \in \mathcal{L}(Z,X)$.

(ii) We have to show that the mapping $X \times \mathbb{R} \to \mathcal{L}(X,Z), (u,\lambda) \mapsto D_uF(u,\lambda)$ is continuous. (We will prove Lipschitz continuity.)

[2] cf. Fredholm Alternative for Boundary Value Problems as discussed in the problem classes

[3] for a reference in German, cf. Werner, Funktionalanalysis, Korollar IV.3.4 (7^{th} edition)

For $(u_1, \lambda_1), (u_2, \lambda_2) \in X \times \mathbb{R}$, we have

$$\begin{aligned}
&\|D_uF(u_1,\lambda_1) - D_uF(u_2,\lambda_2)\|_{\mathcal{L}(X,Z)} \\
&\quad = \sup_{h\in X,\, \|h\|_{C^2([0,1])}=1} \|D_uF(u_1,\lambda_1)[h] - D_uF(u_2,\lambda_2)[h]\|_{C([0,1])} \\
&\quad = \sup_{h\in X,\, \|h\|_{C^2([0,1])}=1} \|(\cos(u_1) - \cos(u_2)) \cdot h\|_{C([0,1])} \le \|\cos(u_1) - \cos(u_2)\|_{C([0,1])} \\
&\quad = \max_{0\le x\le 1} |\cos(u_1(x)) - \cos(u_2(x))| \le \max_{0\le x\le 1} \left| \int_{u_1(x)}^{u_2(x)} |\sin(t)|\, \mathrm{d}t \right| \le \|u_1 - u_2\|_{C([0,1])} .
\end{aligned}$$

The IFT now states that there exist open neighborhoods $U \subseteq X$ of 0, $J \subseteq \mathbb{R}$ of 0 and a continuous function $\hat{u} : J \to U$ with the property that

$$u \in U, \lambda \in J, \quad F(u,\lambda) = 0 \qquad \Leftrightarrow \qquad \lambda \in J, \quad u = \hat{u}(\lambda).$$

Property $(\diamondsuit)$ shows that $\hat{u}(\lambda) \neq 0$ for $\lambda \in J \setminus \{0\}$, and choosing $\delta > 0$ with $(-\delta, \delta) \subseteq J$ and setting $u_\lambda := \hat{u}(\lambda)$, the proof is complete. $\square$