Problem 11 (Differentiability of an integral operator)

Let $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ an open subset, $2 \leq p < \frac{2n}{n-2}$ for $n \geq 3$ and $2 \leq p < \infty$ else. Prove that the following map is continuously Fréchet differentiable

$$ F : H^1_0(\Omega) \to \mathbb{R}, \quad F(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{p} \int_{\Omega} |u|^p \, dx $$

with derivative $F'(u)[h] = \int_{\Omega} \nabla u \cdot \nabla h \, dx - \int_{\Omega} |u|^{p-2} uh \, dx$ where $u, h \in H^1_0(\Omega)$.

**Hint:** By choice of $p$, the continuous Sobolev embedding $H^1_0(\Omega) \subseteq L^p(\Omega)$ holds, i.e. there exists $C > 0$ with the property that $\|u\|_{L^p(\Omega)} \leq C \|u\|_{H^1_0(\Omega)}$ for every $u \in H^1_0(\Omega)$.

**Solution**

We discuss separately $F_0, F_1 : H^1_0(\Omega) \to \mathbb{R}$,

$$ F_0(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx, \quad F_1(u) := \frac{1}{p} \int_{\Omega} |u|^p \, dx. $$

**Continuous Fréchet differentiability of $F_0$:**

For $u, h \in H^1_0(\Omega)$, we calculate directly:

$$ F_0(u + h) - F_0(u) = \frac{1}{2} \int_{\Omega} \nabla (u + h) \cdot \nabla (u + h) - \nabla u \cdot \nabla u \, dx $$

$$ = \int_{\Omega} \nabla u \cdot \nabla h + \frac{1}{2} |\nabla h|^2 \, dx $$

and hence, using the norm $\|h\|^2_{H^1(\Omega)} := \int_{\Omega} |\nabla h|^2 + h^2 \, dx$,

$$ \left| F_0(u + h) - F_0(u) - \int_{\Omega} \nabla u \cdot \nabla h \, dx \right| = \frac{1}{2} \int_{\Omega} |\nabla h|^2 \, dx \leq \frac{1}{2} \|h\|^2_{H^1(\Omega)} = o(\|h\|_{H^1(\Omega)}); $$
Let \( u, h \) be elements of \( H^1(\Omega) \). We conclude that \( F_0 \) is Fréchet differentiable on \( H^1(\Omega) \) with \( F_0'(u)[h] = \int_\Omega \nabla u \cdot \nabla h \, dx \) for \( u, h \in H^1_0(\Omega) \).

To see (even Lipschitz) continuity of \( F_0' \), we estimate for \( u, v \in H^1_0(\Omega) \)

\[
\|F_0'(u) - F_0'(v)\|_{L(H^1(\Omega), \mathbb{R})} = \sup_{h \in H^1_0(\Omega), \|h\|_{H^1(\Omega)} = 1} |F_0'(u)[h] - F_0'(v)[h]| \\
\leq \sup_{h \in H^1_0(\Omega), \|h\|_{H^1(\Omega)} = 1} \int_\Omega |\nabla (u - v) \cdot \nabla h| \, dx \\
\leq \sup_{h \in H^1_0(\Omega), \|h\|_{H^1(\Omega)} = 1} (\|u - v\|_{H^1(\Omega)} \|h\|_{H^1(\Omega)}) = \|u - v\|_{H^1(\Omega)}.
\]

**Continuous Fréchet differentiability of \( F_1 \):**

We intend to apply the result of Problem 10 (a) and prove continuous Gâteaux differentiability.

**Step 1: Gâteaux differentiability of \( F_1 \)**

Let \( u, h \in H^1_0(\Omega) \). By definition of Gâteaux differentiability, we have to prove that

\[
\mathbb{R} \to \mathbb{R}, \quad \tau \mapsto F_1(u + \tau h) = \frac{1}{p} \int_\Omega |u + \tau h|^p \, dx
\]

is differentiable at \( \tau = 0 \) with derivative \( \int_\Omega |u|^{p-2} u h \, dx \). This will be achieved via a standard consequence of dominated convergence (differentiation under the integral sign).

We let \( f(\tau, x) := \frac{1}{p} |u(x) + \tau h(x)|^p \) for \( x \in \Omega, \tau \in \mathbb{R} \). Then, for \( x \in \Omega \), \( f \) is differentiable w.r.t. \( \tau \), and \( \frac{\partial f}{\partial \tau}(\tau, x) = |u(x) + \tau h(x)|^{p-2}(u(x) + \tau h(x))h(x) \). Moreover, for \(-1 \leq \tau \leq 1\), this expression is majorized by

\[
\left| \frac{\partial f}{\partial \tau}(\tau, x) \right| \leq |u(x) + \tau h(x)|^{p-1}|h(x)| \leq (|u(x)| + |h(x)|)^p \leq 2^p(|u(x)|^p + |h(x)|^p),
\]

which is integrable due to the Sobolev embedding given in the hint. Dominated convergence now gives, for \(-1 < \tau < 1\), existence of the derivative

\[
\frac{d}{d\tau} (F_1(u + \tau h)) = \frac{d}{d\tau} \int_\Omega f(\tau, x) \, dx = \int_\Omega \frac{\partial f}{\partial \tau}(\tau, x) \, dx = \int_\Omega |u + \tau h|^{p-2}(u + \tau h) h \, dx.
\]

Moreover, setting \( \tau = 0 \) and for fixed \( u \in H^1_0(\Omega) \), the operator \( H^1_0(\Omega) \to \mathbb{R}, \ h \mapsto \int_\Omega |u|^{p-2} u h \, dx \) is linear and continuous. (The latter is a consequence of the continuous Sobolev embedding in the hint.) Hence, we have Gâteaux differentiability of \( F_1 \) with Gâteaux derivative

\[
dF_1(u)[h] = \left. \frac{d}{d\tau} \right|_{\tau=0} (F_1(u + \tau h)) = \int_\Omega |u|^{p-2} u h \, dx.
\]

**Step 2: Continuity of the Gâteaux derivative of \( F_1 \)**
We let \( u, v_n \in H^1_0(\Omega) \) with \( \|v_n\|_{H^1_0(\Omega)} \to 0 \) as \( n \to \infty \). By the Sobolev embedding in the hint, we also have \( \|v_n\|_{L^p(\Omega)} \to 0 \) as \( n \to \infty \).

The proof of the Riesz-Fischer Theorem shows that we may pass to a subsequence with \( v_{n_k}(x) \to 0 \) and \( |v_{n_k}(x)| \leq \varphi(x) \) for almost all \( x \in \Omega \) and for some function \( \varphi \in L^p(\Omega) \). We then have almost everywhere on \( \Omega \)

\[
\|u + v_{n_k}\|^{p-2}(u + v_{n_k}) - |u|^{p-2}u \leq \left( |(u)| + |\varphi| \right)^{p-1} + |u|^{p-1} \frac{p}{p-1} \\
\leq 2^{\frac{p}{p-1}} \cdot \left( |(u)| + |\varphi| \right)^p + |u|^p \in L^1(\Omega),
\]

hence by dominated convergence

\[
\int_\Omega \|u + v_{n_k}\|^{p-2}(u + v_{n_k}) - |u|^{p-2}u \frac{p}{p-1} dx \to 0 \quad \text{as } k \to \infty.
\]

The argument can be repeated when replacing \((v_n)_{n \in \mathbb{N}}\) by an arbitrary subsequence, yielding a sub-subsequence with the property shown above. Thus,

\[
\int_\Omega \|u + v_n\|^{p-2}(u + v_n) - |u|^{p-2}u \frac{p}{p-1} dx \to 0 \quad \text{as } n \to \infty.
\]

Now, we can estimate the norms using Hölder’s inequality

\[
\|dF_1(u + v_n) - dF_1(u)\|_{\mathcal{L}(H^1_0(\Omega), \mathbb{R})} = \sup_{h \in H^1_0(\Omega), \|h\|_{H^1(\Omega)} = 1} |dF_1(u + v_n)[h] - dF_1(u)[h]|
\]

\[
\leq \sup_{h \in H^1_0(\Omega), \|h\|_{H^1(\Omega)} = 1} \left( \int_\Omega \|u + v_n\|^{p-2}(u + v_n)h - |u|^{p-2}uh \right) dx
\]

\[
\leq \left[ \sup_{h \in H^1_0(\Omega), \|h\|_{H^1(\Omega)} = 1} \|h\|_{L^p(\Omega)} \right] \cdot \left( \int_\Omega \|u + v_n\|^{p-2}(u + v_n) - |u|^{p-2}u \frac{p}{p-1} dx \right)^{\frac{p-1}{p}}
\]

\[
\to 0 \quad \text{as } n \to \infty
\]

since the supremum is finite due to the Sobolev embedding from the hint. This proves continuity, and together with Problem 10 (a) we have continuous Fréchet differentiability of \( F_1 \) and \( F \). \( \square \)
Problem 12 (Directional derivatives)

On the Banach space $L^\infty(\mathbb{R})$, we consider the map $F : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$, $F(u) := |u|^\frac{3}{2}$. Let $u_0 := 1_{[0,1]} \in L^\infty(\mathbb{R})$. For $h \in L^\infty(\mathbb{R})$, prove that the directional derivative

$$\lim_{\tau \to 0} \frac{F(u_0 + \tau h) - F(u_0)}{\tau}$$

exists if and only if $h(x) = 0$ for almost all $x \in \mathbb{R} \setminus [0,1]$. Conclude that $F$ is not Gâteaux differentiable in the point $u_0$.

Solution

(i) First, we consider $h \in L^\infty(\mathbb{R})$ which does not satisfy $h(x) = 0$ a.e. on $\mathbb{R} \setminus [0,1]$.

By assumption, there exists such $\delta > 0$ that the set $N := \{|h| \geq \delta\} \setminus [0,1]$ has positive measure. Then for almost every $x \in N$, we have $u_0(x) = 0$ and $|h(x)| \geq \delta$ and estimate as follows for $\tau \neq 0$:

$$\left| \frac{(F(u_0 + \tau h))(x) - (F(u_0))(x)}{\tau} \right| = \frac{|u_0(x) + \tau h(x)|^\frac{1}{2} - |u_0(x)|^\frac{1}{2}}{\tau} = \frac{|h(x)|^\frac{1}{2}}{\tau} \geq \sqrt{\delta} |\tau|^{-\frac{1}{2}}.$$

Thus, as $\tau \to 0$,

$$\left\| \frac{F(u_0 + \tau h) - F(u_0)}{\tau} \right\|_{L^\infty(\mathbb{R})} \geq \left\| \frac{F(u_0 + \tau h) - F(u_0)}{\tau} \right\|_{L^\infty(N)} \geq \sqrt{\delta} |\tau|^{-\frac{1}{2}} \to \infty,$$

and hence the directional derivative in direction $h$ does not exist.

(ii) Now, we let $h \in L^\infty(\mathbb{R})$ with $h(x) = 0$ a.e. on $\mathbb{R} \setminus [0,1]$. We will prove that the directional derivative exists and is given by

$$\lim_{\tau \to 0} \frac{F(u_0 + \tau h) - F(u_0)}{\tau} = \frac{1}{2} h.$$

We only discuss the case $\|h\|_{L^\infty(\mathbb{R})} > 0$. For $\tau \in \mathbb{R}$ with $0 < |\tau| < \frac{1}{2\|h\|_{L^\infty(\mathbb{R})}}$, we have

$$\left\| \frac{F(u_0 + \tau h) - F(u_0)}{\tau} - \frac{1}{2} h \right\|_{L^\infty(\mathbb{R})} = \left\| \frac{|u_0 + \tau h|^\frac{1}{2} - |u_0|^\frac{1}{2} - \frac{1}{2} h}{\tau} \right\|_{L^\infty([0,1])} = \text{ess sup}_{0 \leq x \leq 1} \sqrt{1 + \tau h(x)} - 1 - \frac{1}{2} \tau h(x) \right\|_{\tau}.$$

We estimate further by Taylor expansion. For almost every $x \in [0,1]$, we have $|\tau h(x)| \leq \frac{1}{2}$, and there exists $\xi(x, \tau) \in (-\frac{1}{2}, \frac{1}{2})$ with

$$\sqrt{1 + \tau h(x)} = 1 + \tau h(x) \cdot \frac{1}{2} + \frac{1}{2} \tau^2 h(x)^2 \cdot \left(-\frac{1}{4}\right) (1 + \xi(x, \tau))^{-\frac{3}{2}},$$
hence we estimate the expression above

\[
\left\| \frac{F(u_0 + \tau h) - F(u_0)}{\tau} - \frac{1}{2} h \right\|_{L^\infty(\mathbb{R})} = \text{ess sup}_{0 \leq x \leq 1} \left| \frac{1}{\tau} \cdot \frac{1}{2} \tau^2 h(x)^2 \cdot \left( -\frac{1}{4} \right) (1 + \xi(x, \tau))^{-\frac{3}{2}} \right| \\
\leq \text{ess sup}_{0 \leq x \leq 1} \left| \frac{1}{8} \tau h(x)^2 \cdot (1 + \xi(x, \tau))^{-\frac{3}{2}} \right| \\
\leq \frac{1}{8} \tau \| h \|_{L^\infty(\mathbb{R})}^2 \cdot \left( \frac{1}{2} \right)^{-\frac{3}{2}} = 2^{-\frac{3}{2}} \| h \|_{L^\infty(\mathbb{R})}^2 \cdot \tau.
\]

We conclude that, as asserted,

\[
\left\| \frac{F(u_0 + \tau h) - F(u_0)}{\tau} - \frac{1}{2} h \right\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as} \quad \tau \to 0.
\]

Finally, we infer that \( F \) is not Gâteaux differentiable at the point \( u_0 \). In fact, Gâteaux differentiability would in particular imply existence of all directional derivatives,

\[
\lim_{\tau \to 0} \frac{F(u_0 + \tau h) - F(u_0)}{\tau} = dF(u_0)[h],
\]

thereby contradicting part (i). □
Problem 13 (An application of the Implicit Function Theorem)

For $\lambda \in \mathbb{R}$, we consider the boundary value problem

\[
\begin{cases}
  u'' - \sin(u) = \lambda e^x & \text{in } (0,1), \\
  u(0) = u(1) = 0.
\end{cases}
\]  

(1)

Show that there exists such $\delta > 0$ that (1) admits a solution $u_\lambda \in C^2([0,1], \mathbb{R}) \setminus \{0\}$ for $0 < |\lambda| < \delta$.

Solution

We consider the Banach spaces $Z := C([0,1], \mathbb{R})$ with norm $\|u\|_{C([0,1])} := \max_{x \in [0,1]} |u(x)|$, $u \in Z$, and $X := \{u \in C^2([0,1], \mathbb{R}) : u(0) = u(1) = 0\}$ with norm $\|u\|_{C^2([0,1])} := \|u\|_{C([0,1])} + \|u'\|_{C([0,1])} + \|u''\|_{C([0,1])}$, $u \in X$.\footnote{$X$ is complete because it is a closed subset of the Banach space $C^2([0,1], \mathbb{R})$ with norm $\|\cdot\|_{C^2([0,1])}$.}

Moreover, we define $F : X \times \mathbb{R} \to Z$, $(F(u, \lambda))(x) := u''(x) - \sin(u(x)) - \lambda e^x$ $(0 \leq x \leq 1)$.

Then every solution $u \in C^2([0,1], \mathbb{R})$ of (1) satisfies $u \in X$ and $F(u, \lambda) = 0$. Conversely, $u \in X$ and $F(u, \lambda) = 0$ implies that $u$ is a $C^2$-solution of (1).

For $\lambda \in \mathbb{R}$, let us note that

\[F(0, \lambda) = 0 \iff \lambda = 0.\]  

(♦)

In order to apply the Implicit Function Theorem in a neighborhood of the zero $(0,0)$ of $F$, we check differentiability of $F$.

Assertion: $F$ is Fréchet differentiable w.r.t $u$, and for $u, h \in X$ and $\lambda \in \mathbb{R}$ the partial (Fréchet) derivative is $D_u F(u, \lambda)[h] = h'' - \cos(u) \cdot h$.

Proof:

For fixed $u \in X$ and $\lambda \in \mathbb{R}$, we let $Ah := h'' - \cos(u) \cdot h$ ($h \in X$). As the cosine is bounded, we have

\[A \in \mathcal{L}(X,Z) \quad \text{with} \quad \|Ah\|_{C([0,1])} \leq \|h\|_{C^2([0,1])} \quad (h \in X).\]

To prove Fréchet differentiability, we estimate for $h \in X$, twice using the fundamental
We are now in a position to verify conditions (i) and (ii) of Theorem III.7 (IFT).

(i) We intend to show that the linear continuous operator $D_u F(0,0) : X \rightarrow Z, h \mapsto h'' - h$ is a homeomorphism.

It is sufficient to prove\(^2\) that $D_u F(0,0)$ is injective, or equivalently, that the homogeneous boundary value problem

$$\begin{cases} h'' - h = 0 & \text{in } (0, 1), \\ h(0) = h(1) = 0 \end{cases}$$

has only got the trivial solution. (Note that the boundary conditions are hidden in the definition of the space $X$.) To do this, we first write down the general solution of the differential equation

$$h(x) = \alpha \cosh(x) + \beta \sinh(x) \quad (0 \leq x \leq 1),$$

and calculate the parameters $\alpha, \beta \in \mathbb{R}$ from the boundary conditions:

$$0 = h(0) = \alpha, \quad 0 = h(1) = \alpha \cosh(1) + \beta \sinh(1),$$

hence $\alpha = \beta = 0$ and $h \equiv 0$. The theorem cited in the footnote then states that $D_u F(0,0)$ is invertible. The Bounded Inverse Theorem\(^3\) implies that $(D_u F(0,0))^{-1} \in \mathcal{L}(Z,X)$.

(ii) We have to show that the mapping $X \times \mathbb{R} \rightarrow \mathcal{L}(X,Z), (u, \lambda) \mapsto D_u F(u, \lambda)$ is continuous. (We will prove Lipschitz continuity.)

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\(^2\)cf. Fredholm Alternative for Boundary Value Problems as discussed in the problem classes

\(^3\)for a reference in German, cf. Werner, Funktionalanalysis, Korollar IV.3.4 (7th edition)
For \((u_1, \lambda_1), (u_2, \lambda_2) \in X \times \mathbb{R}\), we have
\[
\|D_u F(u_1, \lambda_1) - D_u F(u_2, \lambda_2)\|_{\mathcal{L}(X, Z)}
= \sup_{h \in X, \|h\|_{C^2([0,1])} = 1} \|D_u F(u_1, \lambda_1)[h] - D_u F(u_2, \lambda_2)[h]\|_{C([0,1])}
\leq \sup_{h \in X, \|h\|_{C^2([0,1])} = 1} \| \cos(u_1) - \cos(u_2)\|_{C([0,1])} \cdot \|h\|_{C^2([0,1])}
= \max_{0 \leq x \leq 1} |\cos(u_1(x)) - \cos(u_2(x))| \leq \max_{0 \leq x \leq 1} \left| \int_{u_1(x)}^{u_2(x)} \sin(t) \, dt \right| \leq \|u_1 - u_2\|_{C([0,1])}.
\]

The IFT now states that there exist open neighborhoods \(U \subseteq X\) of \(0\), \(J \subseteq \mathbb{R}\) of \(0\) and a continuous function \(\hat{u} : J \to U\) with the property that
\[u \in U, \lambda \in J, \quad F(u, \lambda) = 0 \quad \iff \quad \lambda \in J, \quad u = \hat{u}(\lambda).\]

Property (\()\) shows that \(\hat{u}(\lambda) \neq 0\) for \(\lambda \in J \setminus \{0\}\), and choosing \(\delta > 0\) with \((-\delta, \delta) \subseteq J\) and setting \(u_\lambda := \hat{u}(\lambda)\), the proof is complete.