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Bifurcation Theory

Problem Sheet 5

Problem 14 (Another application of the Implicit Function Theorem)

Let $n \in \mathbb{N}$. Consider a smooth bounded domain $\Omega \subseteq \mathbb{R}^n$ and, for $\varepsilon > 0$, the boundary value problem

$$(\clubsuit)_\varepsilon \qquad \begin{cases} -\Delta u = u^3 & \text{in } \Omega, \\ u \equiv \varepsilon & \text{on } \partial\Omega. \end{cases}$$

Prove that there exists $\varepsilon_0 > 0$ with the property that problem $(\clubsuit)_\varepsilon$ admits a classical solution $u_\varepsilon \in C^2(\overline{\Omega})$ for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

Problem 15 (Choosing the proper function space)

Let $n \in \mathbb{N}$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. In the previous exercise class, we introduced such pairs of spaces X, Z containing functions on Ω with values in $\mathbb{R}$ that the mapping

$$(\spadesuit) \qquad (-\Delta)^{-1} : \; X \to Z, \; f \mapsto u \qquad \text{where} \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega \end{cases}$$

is a linear homeomorphism.

(a) Show that this cannot hold for $Z := H_0^3(\Omega)$ and $X := H^1(\Omega)$.

(b) Construct a counterexample in the case $n = 1$, Ω being a bounded open interval, which shows that $(\spadesuit)$ is in general not well-defined when choosing $X := Z := C_0^\infty(\Omega)$.