Bifurcation Theory

Problem Sheet 6

Problem 16 (The Energy Method revisited)

We return to Remark III.11 (b) and Theorem II.3.

Let $T > 0$, $j \in \mathbb{N}_0$ and $g \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ as in Theorem II.3. For $n \in \mathbb{N}$, consider solutions $u_n \in C^2([0, T])$ and $\lambda_n \in \mathbb{R}$ of

\[\begin{aligned}
-u_n'' &= g(u_n, \lambda_n) \quad \text{in (0, T)}, \\
\quad u_n(0) &= u_n(T) = 0
\end{aligned}\]  

where $u_n$ is $j$-nodal, $u_n > 0$ on $\left(0, \frac{T}{j+1}\right)$ and $\|u_n\|_{C^2([0,T])} \to 0$, $\lambda_n \to \lambda_j^*$.

Prove that $(v_n)_{n \in \mathbb{N}}$ where $v_n := \frac{u_n}{\|u_n\|_{C^2([0,T])}}$ converges in $C^2([0, T])$ and determine the limit. You can proceed as follows:

(a) Prove that there exists a constant $C > 0$ with $\|v_n\|_{C^3([0,T])} \leq C$ for every $n \in \mathbb{N}$. Conclude that the sets $K_j := \left\{v_n^{(j)} : n \in \mathbb{N}\right\} \subseteq C([0, T]), j = 0, 1, 2$, are bounded and equicontinuous.

(b) Deduce that $(v_n)_{n \in \mathbb{N}}$ has a $C^2([0, T])$-convergent subsequence with limit $\varphi_j \in C^2([0, T])$.

Hint: Use part (a) and the Theorem of Arzelà-Ascoli.

(c) To prove the assertion, derive the boundary value problem which is solved by $\varphi_j$.

Problem 17 (The Crandall-Rabinowitz Theorem in finite dimensions)

(a) Determine all bifurcation points $(0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}$ of the nonlinear system

\[\begin{aligned}
\sin(x_1 + \lambda x_2) &= x_1, \\
\cos(\lambda x_1 + x_2) &= 1 + x_1
\end{aligned}\]  

(2)
(b) Let \( A \in \mathbb{R}^{n \times n} \) be symmetric, and \( J \in \mathbb{R}^{n \times n} \). For \( x \in \mathbb{R}^n \) and \( \lambda \in \mathbb{R} \), we study the equation
\[
Ax = \lambda x + |x|^2 Jx.
\] (3)

Discuss the existence of nontrivial solutions in a neighborhood of \((0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}\) if
(i) \( \lambda_0 \) is not an eigenvalue of \( A \),
(ii) \( \lambda_0 \) is a simple eigenvalue of \( A \).

Problem 18 (The Crandall-Rabinowitz Theorem for an ODE)

Let \( g \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \), \((x, z, \lambda) \mapsto g(x, z, \lambda)\) be 2\( \pi \)-periodic in \( x \) with
\[
g(x, 0, \lambda) = 0, \quad g_z(x, 0, \lambda) = 0, \quad g_{z\lambda}(x, 0, \lambda) = 0 \quad \text{for all} \quad x \in \mathbb{R}, \lambda \in \mathbb{R}.
\]

In order to find nontrivial 2\( \pi \)-periodic solutions \( u \in C^2(\mathbb{R}) \) of the ODE
\[
-u'' = \lambda u + g(x, u, \lambda) \quad \text{on} \quad \mathbb{R}
\] (4) in a neighborhood of \((u_0, \lambda_0) = (0, 0)\), proceed as follows:

(a) Let \( F : C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R} \to C^2_{\text{per}}(\mathbb{R}) \), \( F(u, \lambda) := u'' + \lambda u + g(\cdot, u, \lambda) \) where
\[
C^k_{\text{per}}(\mathbb{R}) := \{ u \in C^k(\mathbb{R}) : u(x) = u(x + 2\pi) \text{ for all } x \in \mathbb{R} \} \quad \text{for } k \in \mathbb{N}_0.
\]

Show that \( F \) is (partially) continuously Fréchet differentiable with respect to \( u \).

(b) Show that \( \ker (F_u(0, 0)) = \text{span} \{1\} \); \( \text{ran} (F_u(0, 0)) = \left\{ z \in C^2_{\text{per}}(\mathbb{R}) : \int_0^{2\pi} z(t) \, dt = 0 \right\} \).

(c) Prove that there exist \( \delta > 0 \) and a continuous branch \((-\delta, \delta) \to C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R}, \ s \mapsto (\hat{u}(s), \hat{\lambda}(s))\) with the property that
\[
\left\{ (\hat{u}(s), \hat{\lambda}(s)) : 0 < |s| < \delta \right\}
\]
collects all nontrivial 2\( \pi \)-periodic solutions of problem (4) in a neighborhood of \((0, 0)\).