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Bifurcation Theory

Problem Sheet 6

Problem 16 (The Energy Method revisited)

We return to Remark III.11 (b) and Theorem II.3.

Let $T > 0$, $j \in \mathbb{N}_0$ and $g \in C^2(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ as in Theorem II.3. For $n \in \mathbb{N}$, consider solutions $u_n \in C^2([0,T])$ and $\lambda_n \in \mathbb{R}$ of

$$\begin{cases} -u_n'' = g(u_n, \lambda_n) & \text{in } (0,T), \\ u_n(0) = u_n(T) = 0 \end{cases} \tag{1}$$

where u_n is j-nodal, $u_n > 0$ on $\left(0, \frac{T}{j+1}\right)$ and $\|u_n\|_{C^2([0,T])} \to 0$, $\lambda_n \to \lambda_j^*$.

Prove that $(v_n)_{n\in\mathbb{N}}$ where $v_n := \frac{u_n}{\|u_n\|_{C^2([0,T])}}$ converges in $C^2([0,T])$ and determine the limit. You can proceed as follows:

- **(a)** Prove that there exists a constant $C > 0$ with $\|v_n\|_{C^3([0,T])} \leq C$ for every $n \in \mathbb{N}$. Conclude that the sets $\mathcal{K}_j := \left\{ v_n^{(j)} : n \in \mathbb{N} \right\} \subseteq C([0,T])$, $j = 0, 1, 2$, are bounded and equicontinuous.
- **(b)** Deduce that $(v_n)_{n\in\mathbb{N}}$ has a $C^2([0,T])$-convergent subsequence with limit $\varphi_j \in C^2([0,T])$.

  Hint: Use part (a) and the Theorem of Arzelà-Ascoli.
- **(c)** To prove the assertion, derive the boundary value problem which is solved by φ_j.

Problem 17 (The Crandall-Rabinowitz Theorem in finite dimensions)

- **(a)** Determine all bifurcation points $(0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}$ of the nonlinear system

$$\begin{cases} \sin(x_1 + \lambda x_2) = x_1, \\ \cos(\lambda x_1 + x_2) = 1 + x_1. \end{cases} \tag{2}$$

(b) Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and $J \in \mathbb{R}^{n\times n}$. For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we study the equation

$$Ax = \lambda x + |x|^2 Jx. \tag{3}$$

Discuss the existence of nontrivial solutions in a neighborhood of $(0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ if

(i) λ_0 is not an eigenvalue of A, **(ii)** λ_0 is a simple eigenvalue of A.

Problem 18 (The Crandall-Rabinowitz Theorem for an ODE)

Let $g \in C^2(\mathbb{R}\times\mathbb{R}\times\mathbb{R}, \mathbb{R}), (x, z, \lambda) \mapsto g(x, z, \lambda)$ be 2π-periodic in x with

$$g(x, 0, \lambda) = 0, \quad g_z(x, 0, \lambda) = 0, \quad g_{z\lambda}(x, 0, \lambda) = 0 \quad \text{for all } x \in \mathbb{R}, \lambda \in \mathbb{R}.$$

In order to find nontrivial 2π-periodic solutions $u \in C^2(\mathbb{R})$ of the ODE

$$-u'' = \lambda u + g(x, u, \lambda) \qquad \text{on } \mathbb{R} \tag{4}$$

in a neighborhood of $(u_0, \lambda_0) = (0, 0)$, proceed as follows:

(a) Let $F : C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R} \to C_{\text{per}}(\mathbb{R}),\ F(u, \lambda) := u'' + \lambda u + g(\,\cdot\,, u, \lambda)$ where

$$C^k_{\text{per}}(\mathbb{R}) := \left\{u \in C^k(\mathbb{R}) :\ u(x) = u(x + 2\pi) \text{ for all } x \in \mathbb{R}\right\} \qquad \text{for } k \in \mathbb{N}_0.$$

Show that F is (partially) continuously Fréchet differentiable with respect to u.

(b) Show that $\ker(F_u(0, 0)) = \text{span}\ \{\mathbb{1}\}$; $\text{ran}\,(F_u(0, 0)) = \left\{z \in C_{\text{per}}(\mathbb{R}) :\ \int_0^{2\pi} z(t)\,\mathrm{d}t = 0\right\}$.

(c) Prove that there exist $\delta > 0$ and a continuous branch $(-\delta, \delta) \to C^2_{\text{per}}(\mathbb{R}) \times \mathbb{R},\ s \mapsto (\hat{u}(s), \hat{\lambda}(s))$ with the property that

$$\left\{(\hat{u}(s), \hat{\lambda}(s)) :\ 0 < |s| < \delta\right\}$$

collects all nontrivial 2π-periodic solutions of problem (4) in a neighborhood of $(0, 0)$.